Panel 1

Last Topic: Integration

$$\int_\gamma f(z)dz = \int_a^b f(z(t))\, z'(t)dt$$

① $\gamma$: parametrize via $z(t) = ?$

Ex $\int_\gamma z^2 dz = \int_0^1 (i + t(1-i))^2 (1-i)dt = \#$

$\gamma$ from $i$ to $1$. $z(t) = i + t(1-i)$, $t \in [0,1]$

Panel 2

Last Theorem: Suppose $f$ is continuous in a domain $D$. Then the following are equivalent:

(1) $f$ has antiderivative $F$ on and near $\gamma$

② $\int_\gamma f(z)dz = F(b) - F(a)$.

③ $\int_\gamma f(z)dz = 0$ for all closed curves $\gamma$!

Panel 3

Ex: $\int_\gamma z^2\,dz$, $\gamma$ line from $i$ to $1$.

$f(z)=z^2$ has antideriv. $F(z)=\frac{1}{3}z^3+\cancel{c}$

$$\Rightarrow \int_\gamma z^2\,dz = F(1)-F(i) = \left(\frac{1}{3}+\cancel{c}\right)-\left(\frac{1}{3}i^3+\cancel{c}\right)$$

$$=\frac{1}{3}(1+i)$$

Panel 4

Ex: Evaluate $\int_\gamma \frac{1}{z}\,dz$ for $|z|=R$, $\gamma$ circle from $1$ to $1$

Antideriv. of $1/z = \mathrm{Log}(z)$ NOT

$$\int_\gamma \frac{1}{z}\,dz = \mathrm{Log}(1)-\mathrm{Log}(1)=0$$

old fashioned: $z(t)=e^{it}$, $t\in[0,2\pi)$

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\cancel{e^{it}}}\cdot i\cancel{e^{it}}\,dt = 2\pi i$$

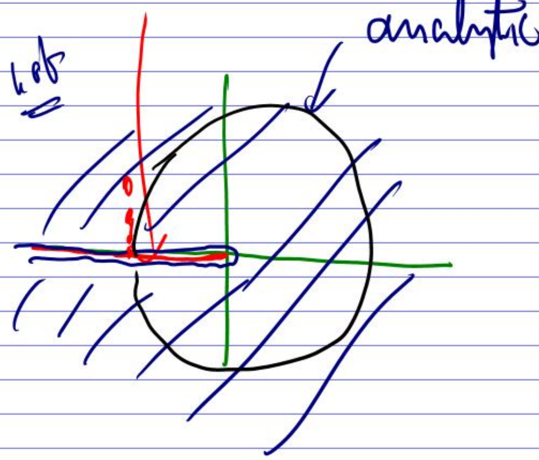
What gives!!!

Problem: Log(z)!!!

Panel 5

Thm: $\text{Log}(z)$ is not analytic for $\{\text{Re}(z) \leq 0\}$

In fact, not even continuous there!

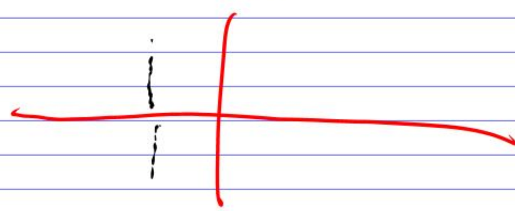



take $z_j = -1 + \frac{i}{n}$

$\lim \text{Log}(z_j) = \pi$

take $z_j = -1 - \frac{i}{n}$

$\lim L(z_j) = -\pi$



Panel 6

$\text{Log}(z)$ is not cont. on $\{\text{Re}(z) \leq 0\}$

Redefine. $\text{Log}(z) = \ln(|z|) + i(\theta), \; -\pi < \theta < \pi$

Thus, $\text{Log}(z)$ is not antiderivative of $\frac{1}{z}$
on the entire unit circle!

Panel 7

Theorem: $C$ is a closed, simple curve and $f$ is analytic inside and on $C$. Then:

$$\int_C f(z)\,dz =$$

Recall Green's Theorem: If $P(x,y)$ and $Q(x,y)$ are continuous in $D$, and all partials exist and are continuous then:

$$\int_C P\,dx + Q\,dy =$$

Panel 8

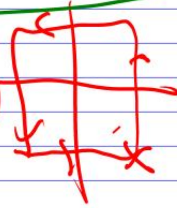
Theorem: $C$ is a closed, simple curve and $f$ is analytic inside and on $C$. Then:

$$\int_C f(z)\,dz = 0$$


Recall Green's Theorem: If $P(x,y)$ and $Q(x,y)$ are continuous in $D$, and all partials exist and are continuous then:

$$\boxed{\int \vec{F}\,d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D Q_x - P_y\,dA}$$

Ex $\displaystyle\int_{\text{square}} z^2+5z+z^3\,dz = 0$, $\displaystyle\int_{|z|=1} e^{z^3}\,dz = 0$



Panel 9

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+idy) =$$

$$= \int u\,dx - v\,dy + i\int u\,dy + v\,dx$$

$P = u,\ Q = -v$ $\qquad$ $P = v\quad Q = u$

$$= \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA + i\iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \iint \cancelto{0}{-v_x - u_y}\,dA + i\iint \cancelto{0}{u_x - v_y}\,dA =$$

Know: $u_y = -v_x,\ u_x = v_y$ $\qquad = 0\ !$ q.e.d.

Panel 10

Cauchy: $f$ analytic + $f'$ continuous inside and on $C$

Then: $\int_C f(z)\,dz = 0$ $\quad\forall$ closed simple curves

Cauchy-Goursat Thm: If $f$ is analytic inside and on $C$ then

$$\int_C f(z)\,dz = 0 \qquad \forall \text{ closed simple curves } C$$

Panel 11

**Augustin-Louis Cauchy**

Augustin-Louis Cauchy around 1840./ Lithography of Zéphirin Belliard after a painting by Jean Roller.

| | |
|---|---|
| **Born** | 21 August 1789<br>Paris, France |
| **Died** | 23 May 1857 (aged 67)<br>Sceaux, France |
| **Residence** | France |
| **Nationality** | French |
| **Fields** | Infinitesimal calculus<br>Complex analysis |

**Edouard Goursat**

Edouard Goursat

| | |
|---|---|
| **Born** | 21 May 1858<br>Lanzac, Lot |
| **Died** | 25 November 1936 |
| **Nationality** | France |
| **Fields** | mathematics |
| **Alma mater** | École Normale Supérieure |
| **Doctoral advisor** | Gaston Darboux |

Panel 12

Cauchy - Goursat Theorem (Improved) no holes

If $f$ is analytic in a simply connected domain $D$. Then

$$\int_C f(z)\,dz = 0 \quad \forall \text{ simple closed curves } C \text{ in } D$$

non-intersecting

Corollary: If $f$ is analytic in a simply connected domain $D$ then $f$ has antiderivative

Proof: $f$ analytic in $D \xRightarrow{\text{Cauchy G.}} \int_\gamma f\,dz = 0$ for closed $\gamma$

$\Rightarrow$ $f$ has antideriv. !!

qed

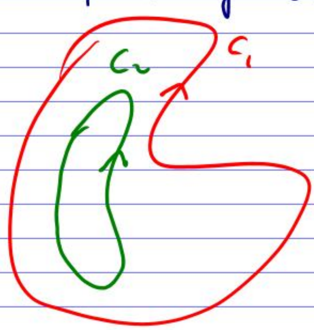
Panel 13

## Cauchy - Goursat Theorem

If $f$ is analytic in a simply connected domain $D$ then

$$\int_C f(z)\,dz = 0 \quad \text{for every closed curve } C \text{ in } D$$

Corollary: If $C_1$ and $C_2$ are two simple closed curves, positively oriented, with $C_1$ inside $C_2$. Then




$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

(Deformation theorem)

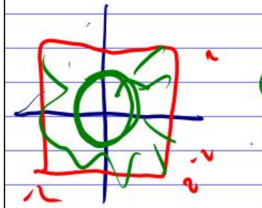
Panel 14

Ex: Find $\int_C e^{z^3}dz$ where $C$ is square of length 4.

$\int_C e^{z^3}dz = 0$ (Cauchy - Goursat)

Ex: Find $\int_C \frac{1}{3z}\,dz$ where $C$ square of side 4.

No Cauchy-Goursat thm applies



Def Thm: $\int_C \frac{1}{3z}\,dz = \int_{|z|=1} \frac{1}{3z}\,dz = \int_0^{2\pi} \frac{1}{3e^{it}}\, ie^{it}dt$

$z = e^{it}$

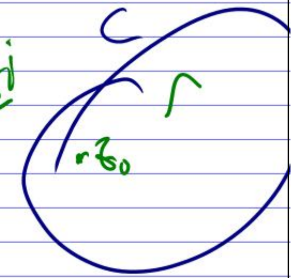
Ex: Find $\int_C \bar{z}\,dz$ where $C$ square of side 4.

$\bar{z}$ is nowhere analytic. Must integrate manually along 4 paths!!!

Panel 15

Cauchy's Integral Formula: f analytic inside and on simple closed curve C, positively oriented. If $z_0$ is any point inside C then

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$



Ex: $\displaystyle\int_{|z|=2} \frac{z^2+3z+1}{z-1}\,dz = \left(f(1)\right)2\pi i = \left(1^2+3(1)+1\right)10\pi i$

$f(z)=z^2+3z+1$

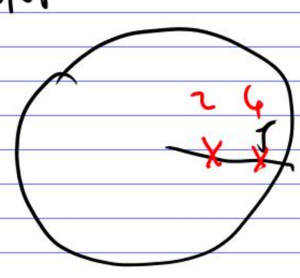
$z=2e^{it}$

Panel 16

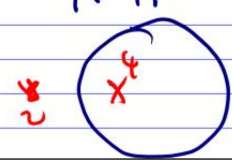
Ex: $\displaystyle\int_{|z|=1}\frac{z}{(z-2)(z-4)}\,dz = 0$ and $\displaystyle\int_{|z|=3}\frac{z}{(z-2)(z-4)}\,dz = \int_{|z|=3}\frac{f(z)}{z-2}\,dz$

$= 2\pi i\, f(2) = 2\pi i\,\frac{2}{2-4} = -2\pi i$

$\displaystyle\int_{|z|=5}\frac{z}{(z-2)(z-4)}\,dz = \int \frac{A}{z-2}\,dz + \frac{B}{z-4}\,dz = \cancel{(A+B)(4\pi i)}$



$\displaystyle\int_{|z-4|=1}\frac{z}{(z-2)(z-4)}\,dz = 2\pi i\, f(4) = 2\pi i\, 2$



Panel 17

Goal: Integration made easy!

$$\int_{|z|=1} \frac{e^z}{z-2}\,dz = 0$$

$$\int_{|z|=1} \frac{e^z}{z-0}\,dz = 2\pi i\, e^0$$

$$\int_{|z|=2} \frac{e^z}{z(z-3)^2}\,dz = 2\pi i\, \frac{e^0}{(-3)^2} = \frac{2\pi i}{9}$$