Panel 1

We defined special functions:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$\log(z) = \ln(r) + i(\theta + 2k\pi) \quad \text{if } z = re^{i\theta}$$

$$\mathrm{Log}(z) = \ln(r) + i\theta, \quad -\pi < \theta \leq \pi$$

Panel 2

If $z(t) = x(t) + iy(t)$, $t \in [a,b]$ is a path in $\mathbb{C}$:

$$\int_a^b z(t)\,dt = \int_a^b x(t)\,dt + i y(t)\,dt$$

← 2 calc 1 problems

If $z(t)$, $t \in [a,b]$ is the parametrization of a path γ in $\mathbb{C}$ and $f:\mathbb{C} \to \mathbb{C}$ a complex function.

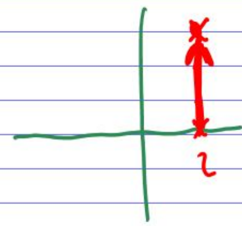
$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

Contour or line integrals

Panel 3

Examples:

$\gamma(t) = 2 + 3it, \quad t \in [0,1]$



$$\Rightarrow \int_0^1 \gamma(t)\,dt = \int_0^1 2 + i\,3t\,dt = 2t\Big|_0^1 + i\,3\,\frac{t^2}{2}\Big|_0^1 = 2 + i\,\frac{3}{2}$$

$\gamma(t) = 2 + 3it, \ t \in [0,1], \quad f(z) = z^2$

$$\Rightarrow \int_\gamma f(z)\,dz = \int_\gamma z^2\,dz = \int (2+3it)^2\,\gamma'(t)\,dt =$$

$$= \int_0^1 (2+3it)^2 (3i)\,dt = 3i \int_0^1 4 + 12it - 9t^2\,dt = \_\_\_$$

Panel 4

$$\cos(3i) = \frac{1}{2}\left(e^{i(3i)} + e^{-i(3i)}\right) =$$

$$= \frac{1}{2}\left(e^{-3} + e^{3}\right) \approx 11.$$

$|\cos(z)|$ unbounded

Panel 5

Analytic : $f(z)$ is analytic at z_0 if f is $\mathbb{C}$-diff'ble at z_0 and all points close-by.

$\mathbb{C}$-diff'ble: $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists $\Leftrightarrow$ CR check

harmonic: $u_{xx} + u_{yy} = 0$

CR : $f(z) = u(x,y) + i v(x,y)$, $u_x = v_y$

$u_y = -v_x$

Entire: analytic everywhere!

Panel 6

If $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u} + \underbrace{2ixy}_{v}$

u, v are harmonic!

If u is harmonic, find v harmonic s.t. $u + iv$ is analytic. v is called harmonic conjugate

Ex: $u(x,y) = x^3 - 3xy^2$ harmonic?

$u_x = 3x^2 - 3y^2$, $u_{xx} = 6x$

$u_y = -6xy$, $u_{yy} = -6x$

yes!

Find harmonic conjugate, i.e.

Panel 7

find V s.t. $u + iV$ is analytic. $(u = x^3 - 3xy^2)$

$\Rightarrow$ CR check: $u_x = V_y$

$$3x^2 - 3y^2 = V_y$$

$$\Rightarrow V = \int 3x^2 - 3y^2 \, dy = 3x^2 y - y^3 + C(x)$$

$$u_y = -V_x$$

$$\Rightarrow \cancel{-6xy} = -(\cancel{6xy} + C'(x))$$

$$\Rightarrow C'(x) = 0, \quad C(x) = c$$

$u = x^3 - 3xy^2$, $V = 3x^2y - y^3 + c$

$u + iV$ is analytic

Panel 8

u has 2 harmonic conjugates v and V

Define $G(x,y) = v(x,y) - V(x,y)$

$$\Rightarrow G_x = v_x - V_x = u_y - u_y \equiv 0$$

$$G_y = v_y - V_y = -u_x + u_x \equiv 0$$

G is a function s.t. $\nabla G \equiv 0$

$\Rightarrow G$ must be constant

#

Panel 9

$u(x,y) = \frac{y}{x^2+y^2}$ is harmonic?

∂_x

$u_x = \frac{-2xy}{(x^2+y^2)^2}$, $u_{xx} = \frac{-2y(x^2+y^2)^2 + 8xy \cdot (x^2+y^2)}{(x^2+y^2)^{4\,3}}$

$u_{xx} = \frac{-2y(x^2+y^2) + 8x^2y}{(x^2+y^2)^3} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$

$u_{yy} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$ $\Rightarrow$ u is harmonic!

$v_x = u_y = \frac{x^2-y^2}{(x^2+y^2)^2}$ $\Rightarrow v = \int \frac{x^2-y^2}{(x^2+y^2)^2}\,dx = \frac{-x}{x^2+y^2} + C(y)$

Panel 10

$v_y = \frac{\partial}{\partial y}\left(\frac{-x}{x^2+y^2} + C(y)\right) = +\frac{2xy}{(x^2+y^2)^2} + C'(y) = -u_x$

$\Rightarrow C'(y) = 0,\ C = c$

$\Rightarrow v = \frac{-x}{x^2+y^2} + c$

$\Rightarrow u + iv = \frac{y}{x^2+y^2} - i\frac{x}{x^2+y^2} + c$ is analytic

Panel 11

$u(x,y) = \frac{y}{x^2+y^2}$ ✓

$u_x = v_y = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow v = \int v_y \, dy = \frac{x}{x^2+y^2} + C$

$u_y = -v_x \Rightarrow C' = 0$

$v(x,y) = \frac{x}{x^2+y^2}$ is harm. conjugate

$f(z) = \frac{y + ix}{x^2+y^2}$ is analytic

$= \frac{i(x-iy)}{z\bar{z}} = \frac{i\bar{z}}{z\bar{z}} = \frac{i}{z}$

Panel 12

$\log(3^i) = \log(|3^i|) + i(\theta + 2k\pi)$

$3^i = e^{\log(3^i)} = e^{i\log(3)} =$

$\theta = \log(3)$

$\log(3^i) = \underline{i(\log(3)} + 2k\pi)$