Panel 1

We defined special functions:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$$

unbdd, periodic

$$\log(z) = \ln(|z|) + i(\theta + 2k\pi)$$

$$\mathrm{Log}(z) = \ln(|z|) + i\theta \quad , \theta = \text{principle angle}$$

$\mathbb{R}$, natural log

Panel 2

$$\mathrm{Log}(i) = \ln(1) + i\frac{\pi}{2} = i\frac{\pi}{2}$$ preferred

$$\log(i) = i\left(\frac{\pi}{2} + 2k\pi\right) = i\frac{\pi}{2}, \ i\frac{5\pi}{2}, \ i\frac{9\pi}{2}, \ldots$$

$$i^{\pi} = e^{\log(i^{\pi})} = e^{\pi \mathrm{Log}(i)} = e^{\pi i \frac{\pi}{2}} = e^{i\pi^2/2}, \ |i^{\pi}| =$$

$$\pi^i = e^{\mathrm{Log}(\pi^i)} = e^{i\,\mathrm{Log}(\pi)} = \cos(\ln(\pi)) + i\sin(\ln(\pi))$$

$$i^i = e^{i\,\mathrm{Log}(i)} = e^{i \cdot i\pi/2} = e^{-\frac{\pi}{2}}$$

$$i^{i^i} \begin{cases} (i^i)^i = i^{(i^2)} = i^{-1} = 1/i = -i \\ i^{(i^i)} = i^{e^{-\pi/2}} = e^{\mathrm{Log}(i^{e^{-\pi/2}})} = e^{e^{-\frac{\pi}{2}}\mathrm{Log}(i)} = e^{i\frac{\pi}{2}e^{-\frac{\pi}{2}}} \end{cases}$$

Panel 3

$$e^{z} = -2 = e^{x}e^{iy} < \begin{array}{l} |-2| = |e^{x}e^{iy}| \Rightarrow 2 = e^{x} \rightarrow x = \ln|2| \\ e^{\ln(2)} e^{iy} = 2e^{iy} = -2 \end{array}$$

$$\underline{y = \pi}$$

$$z = \ln|2| + i\pi$$

$$\cos(z) = 3 = \frac{e^{iz} + e^{-iz}}{2} = 3$$

$$e^{iz} + e^{-iz} = 6 \Leftrightarrow e^{ix-y} + e^{-ix+y} = 6$$

$$(\cos(x) + i\sin(x))e^{-y} + (\cos(-x) + i\sin(-x))e^{y} = 6$$

$$e^{-y}(\cos(x) + i\sin(x)) + e^{y}(\cos(x) - i\sin(x)) = 6$$

$$\cos(x)(e^{-y} + e^{y}) + i\sin(x)(e^{-y} - e^{y}) = 6$$

Panel 4

$$\cos(x)(e^{-y} + e^{y}) + i\sin(x)(e^{-y} - e^{y}) = 6$$

$$\sin(x)(e^{-y} - e^{y}) = 0 \Rightarrow \cancel{y = 0}$$

$$x = 0$$

$$\Rightarrow e^{-y} + e^{y} = 6 \Rightarrow \frac{e^{y} + e^{-y}}{2} = 3$$

$$\Rightarrow \cosh(y) = 3 \Rightarrow y = \cosh^{-1}(3) = 1.762747$$

$$\cos(z) = 3 \Leftrightarrow z = i\,1.762747$$

checks out!!!

Panel 5

Integrals:

Integration in Complex Analysis is very important and leads to beautiful, profound theorems with elegant proofs:

Pure mathematics is, in its way, the poetry of logical ideas. ~Albert Einstein

Before we integrate....

Panel 6

Complex Parametric Functions e.g. $(\cos(t), \sin(t))$

Def: A parametric function $r(t) = (x(t), y(t))$ is a function from $\mathbb{R}$ to $\mathbb{R}^2$. A complex parametric function $w(t) = x(t) + i y(t)$ is a function from $\mathbb{R} \to \mathbb{C}$

Ex: $z(t) = (3+i) + (1-i)t$ is a line in $\mathbb{C}$ through $(3+i)$, 4

$z(t) = \cos(t) + i \sin(t)$ is a circle in $\mathbb{C}$

Panel 7

Derivatives of Complex parametric functions:

Def: If $z(t) = x(t) + i\,y(t)$ then $\frac{dz}{dt} = x'(t) + i\,y'(t)$

Ex: If $z(t) = 5\cos(3t) + 5i\sin(3t)$, find $z'(t)$

$z'(t) = -15\sin(3t) + i\,15\cos(3t)$

either

or: $z(t) = 5e^{i3t}$

way!

$\Rightarrow z'(t) = 5i \cdot 3e^{i3t} = 15i\,e^{i3t}$

Panel 8

Integrals of Complex Parametric Functions

Def: If $z(t) = x(t) + i\,y(t)$ then

$$\int z(t)\,dt = \int x(t)\,dt + i\int y(t)\,dt$$

Ex: $\int_0^1 (1+it)^2\,dt = \int_0^1 1 + 2it - t^2\,dt =$

$\int_0^1 1 - t^2\,dt + i\int_0^1 2t\,dt$ #

Panel 9

Ex: $\int_0^\pi e^{it}\,dt = \int_0^\pi \cos(t) + i\sin(t)\,dt =$

$= \sin(t)\Big|_0^\pi - i\cos(t)\Big|_0^\pi =$

$= \cancel{\sin(\pi)} - \cancel{\sin(0)} - i\left(\cos(\pi) - \cos(0)\right) =$

$-i(-1-1) = \underline{2i}$

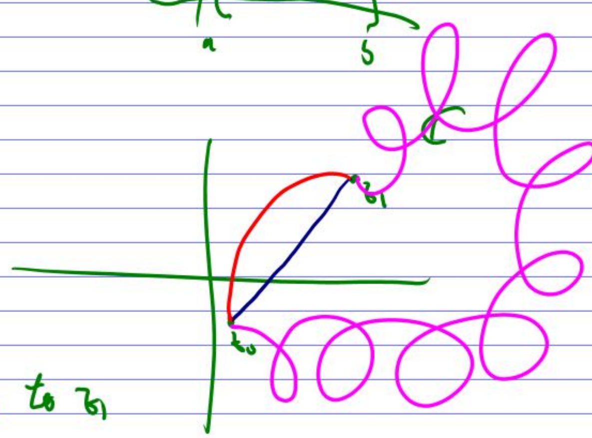
$\int_0^\pi e^{it}\,dt = \frac{1}{i} e^{it}\Big|_0^\pi = -i(e^{\pi i} - e^0) = -i(-1-1) = \underline{2i}$

Panel 10

How to define $\int_{z_1}^{z_2} f(z)\,dz$

$\mathbb{R}$: $\int_a^b f(t)\,dt$

$\mathbb{C}$: $\int_{z_0}^{z_1} f(z)\,dz$



Many paths from z_0 to z_1

Panel 11

Contour Integral:

Suppose $z_1, z_2 \in \mathbb{C}$ and γ is a path from z_1 to z_2 with some parametrization $z(t)$, $t \in [a,b]$. Then:

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,\frac{dz}{dt}\,dt = \int_a^b f(z(t))\,z'(t)\,dt$$

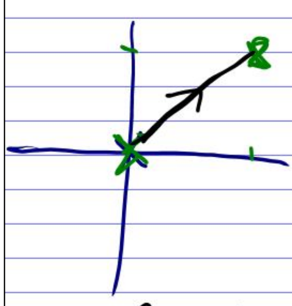
... before we go on $\Rightarrow$

Panel 12

Ex: Let γ be the straight line from $z_0 = 0$ to $z_1 = 1+i$.

Compute $\displaystyle\int_\gamma z^2\,dz = \int_0^1 \big(\underbrace{(1+i)t}_{z}\big)^2 \cdot (1+i)\,dt =$

$$\int_0^1 (1+i)^3\, t^2\,dt = (1+i)^3 \cdot \frac{1}{3} t^3 \Big|_0^1 = \frac{(1+i)^3}{3}$$



$\gamma : z(t) = 0 + (1+i)t \,, \quad t \in [0,1]$

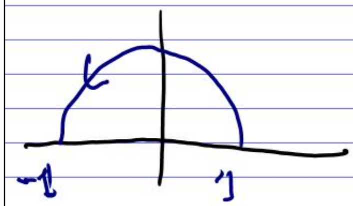
$= (1+i)t$

line from A to B is $\ell(t) = A + (B-A)t$

$t \in [0,1]$

Panel 13

Find $\int_\gamma z^2+3z\,dz$. γ is upper half of unit circle.



$\gamma: z(t)=e^{it}, \quad t\in[0,\pi]$

$dz = ie^{it}\,dt$

$$\int_\gamma z^2+3z\,dz = \int_0^\pi \left[(e^{it})^2+3(e^{it})\right](ie^{it})\,dt$$

$$= i\int_0^\pi e^{3it}+3e^{2it}\,dt = \cancel{i}\left(\frac{1}{3\cancel{i}}e^{3it}+\frac{3}{2\cancel{i}}e^{2it}\right)\Bigg|_0^\pi$$

$$\left(\frac{1}{3}e^{3\pi i}+\frac{3}{2}e^{2\pi i}\right)-\left(\frac{1}{3}+\frac{3}{2}\right) =$$

$$-1/3+3/2-1/3-3/2 = \boxed{-2/3}$$