Panel 1

Complex Limits: inf. many directions to test; pick approach curves to z_0 get different limits.
$\Rightarrow$ limit DNE.

Given $\varepsilon > 0$ there is $\delta > 0$ s.t.
$|z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$

$\mathbb{C}$-differentiable

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

Analytic function: f is analytic at z_0 if f is $\mathbb{C}$-diff'ble at z_0 and at every point near z_0

Panel 2

l'Hopital's Theorem: If f, g are analytic, $f(z_0) = g(z_0) = 0$ then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)} = \frac{f'(z_0)}{g'(z_0)}$$

if $g'(z_0) \neq 0$

CR Equations: $f(z) = u + iv$ is $\mathbb{C}$-diff'ble iff

$$u_x = v_y$$

$$u_y = -v_x$$

Converse is (usually) true

$f' = u_x + i v_x$

Harmonic functions: $u: \mathbb{R}^2 \to \mathbb{R}$ harmonic if

$$\Delta u = u_{xx} + u_{yy} = 0$$

Panel 3

Entire functions: $f : \mathbb{C} \to \mathbb{C}$ s.t. f is analytic $\forall z \in \mathbb{C}$

Relation between analytic and harmonic fctns

Thm: If f is analytic, then both u, v are harmonic

Panel 4

Thus:

f analytic and real-valued

f analytic and $\bar{f}$ analytic

f analytic and $|f(z)|$ constant

$\Rightarrow$ f is const.

Panel 5

Complex Concepts

Limits:

Continuity:

$\mathbb{C}$ - Diff ble:

Analytic

Harmonic

Panel 6

CR: $u_x = v_y$

$u_y = -v_x$

Theorems:

f analytic & $f'(z) \equiv 0$
f analytic, $|f(z)| \equiv c$
f & $\bar{f}$ both analytic
f analytic, real valued

$\Bigg\}$ f is const

Panel 7

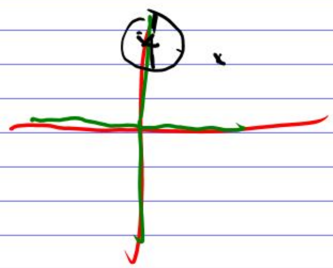
Ex: Is $f(z) = \underbrace{x^3 + 3xy^2}_{u} + i\underbrace{(y^3 + 3x^2y)}_{v}$ diffble anywhere?
Is it analytic anywhere?

$u_x = 3x^2 + 3y^2 = v_y = 3y^2 + 3x^2$

$u_y = 6xy = -v_x = 6xy$

$\mathbb{C}$-diffble if $6xy = -6xy \Rightarrow x = 0$ or $y = 0$

Not analytic anywhere!



Panel 8

Ex: Which of these are harmonic functions:

$u(x,y) = x^3 - 3xy^2$ YES

$u_x = 3x^2 - 3y^2$, $u_{xx} = 6x$

$u_y = -6xy$, $u_{yy} = -6x$

$v(x,y) = 3x^2y + y^3$ NOT

$u_x = 6xy$, $u_{xx} = 6y$

$u_y = 3x^2 + 3y^2$ $u_{yy} = 6y$

$T(x,y) = e^{-y}\sin(x)$ YES

HW

Panel 9

Theorem: If f is analytic in a domain D and
$f(z) = u(x,y) + iv(x,y)$ then
both u, v are harmonic

Def: If u is harmonic, and v is another harm. function s.t. $u+iv$ is analytic.
then v is called harmonic conjugate to u.

Ex: $u(x,y) = x^2 - y^2 + i\,\underbrace{2xy} = z^2$.
Thus: $2xy$ is harm. conj. to x^2-y^2

Panel 10

Thm: If $u(x,y)$ is harmonic in a 'special' domain D
then u has a harmonic conjugate

Ex: $u(x,y) = x^3 - 3xy^2$ is harmonic. Find harmonic conjugate!
$\Rightarrow f = u+iv$ analytic $\Rightarrow$ CR

$u_x = 3x^2 - 3y^2 = v_y$

$\Rightarrow v = \int 3x^2 - 3y^2\, dy = \underline{3x^2y - y^3} + C(x)$
$\Rightarrow v_x = 6xy + C'(x) = -u_y = +(+6xy)$
$\Rightarrow \underline{C'(x) = 0}$

Thus: $x^3 - 3xy^2 + i(3x^2y - y^3) \quad (= z^3)$

Panel 11

$\underline{Ex}$: $u(x,y) = e^{-y}\sin(x)$. Find $v(x,y)$ such that $f(z) = u(x,y) + i\,v(x,y)$ is analytic. Find $f'(z)$???

$u_x = e^{-y}\cos(x)$, $u_{xx} = -e^{-y}\sin(x)$

$u_y = -e^{-y}\sin(x)$, $u_{yy} = e^{-y}\sin(x)$ $\Big\} \Delta u = 0$ ✓

$u_x = e^{-y}\cos(x) = v_y \Rightarrow \underline{v = -e^{-y}\cos(x) + C(x)}$

$v_x = e^{-y}\sin(x) + C'(x) = -u_y \Rightarrow C'(x) = 0$

$f = u + iv = e^{-y}\cos(x) + i(-e^{-y}\cos(x))$

$f'(z) = u_x + iv_x = e^{-y}\sin(x) + i\,e^{-y}$

Panel 12

$f(z) = z = x + iy$

$u_x = 1$, $u_y = 0$

$v_x = 0$, $v_y = 1$

$f'(z) = 1 = u_x + iv_x$

$z^2 = x^2 - y^2 + 2ixy$

$u_x = 2x$ $\quad v_x = 2y$

$u_y = -2y$ $\quad v_y = 2x$

$f' = 2z = 2(x + iy) =$

$= u_x + i\,v_x$