Panel 1

Last Time

$\mathbb{C}$-diff'ble: $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

$\mathbb{C}$-diff'ble $\Rightarrow$ continuity

Functions in z are usually $\mathbb{C}$-diff'ble with rules such as Quotient, Chain, Product, Power, ...

CR Equations: $f(z) = u(x,y) + i v(x,y)$

If f is $\mathbb{C}$ diff'ble, then

$$u_x = v_y$$

$$u_y = -v_x$$

Private Freehand 1

Panel 2

$$\lim_{z \to -2i} \frac{z^2+4}{z^2+iz+2} \quad \text{(WOLFRAM)}$$

$$\to \lim_{z \to -2i} \frac{z^2+4}{z^2+iz+2} = \frac{(z+2i)(z-2i)}{(z+2i)(z-i)}$$

$$\lim_{z \to -2i} \frac{z-2i}{z-i} = \frac{-4i}{-3i}$$

$$\boxed{= 4/3}$$

Panel 3

$$\lim_{h \to 0} \frac{f(z+h)-f(z)}{h} \qquad h = h_1 + ih_2 \qquad 2z^2-3z+4$$

$$\lim_{h \to 0} \frac{2(z+h)^2 - 3(z+h)+4 - (2z^2-3z+4)}{h}$$

$$\lim_{h \to 0} \frac{2(z^2+2zh+h^2) - 3z-3h+4-2z^2+3z-4}{h}$$

$$\lim_{h \to 0} \frac{\cancel{2z^2} + 4zh + 2h^2 - \cancel{3z} - 3h + \cancel{4} - \cancel{2z^2} + \cancel{3z} - \cancel{4}}{h}$$

$$\lim_{h \to 0} = \frac{4zh+2h^2-3h}{h} = \frac{2h^2+4zh-3h}{h}$$

$$\frac{\cancel{h}(2h+4z-3)}{\cancel{h}} \qquad 4z-3$$

Panel 4

$f(z) = 2x + iy = \frac{3}{2}z + \frac{1}{2}\bar{z}$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{2x + iy - (2x_0 + iy_0)}{x + iy - (x_0 + iy_0)} =$$

$$= \lim_{(x,y) \to (x_0,y_0)} \frac{2(x - x_0) + i(y - y_0)}{(x - x_0) + i(y - y_0)}$$

$x = x_0,\ y \to y_0$: $\lim_{y \to y_0} \frac{\cancel{i}\cancel{(y-y_0)}}{\cancel{i}\cancel{(y-y_0)}} = 1$ now

$y = y_0,\ x \to x_0$: $\lim_{x \to x_0} \frac{2(x - x_0)}{(x - x_0)} = 2$

Panel 5

$f(z) = \underbrace{2x}_{u(x,y)} + i\underbrace{y}_{v(x,y)} = u + iv$

$u_x \quad -v_x$

$u_y \quad v_y$

$u_x = 2,$

$v_y = 1$

$f(z) = x^2 + y^2 \in \mathbb{R}$

Pr

Panel 6

Show: If $f(z)$ is real and $\mathbb{C}$-diff'ble $\Rightarrow f = c$

$f(z) = u(x,y) + i v(x,y)$

f is real-valued $\Rightarrow v(x,y) \equiv 0$

$\Rightarrow u_x = v_y = 0$

$u_y = -v_x = 0$

$u(x,y)$ is real valued with $u_x = u_y = 0 \quad \forall (x,y)$

$\Rightarrow u$ is constant

$\Rightarrow f(z) = u + iv = c$

Panel 7

Theorem: l'Hopital's Rule

Prove that if $f(z_0) = g(z_0) = 0$ and $f'(z_0)$, $g'(z_0)$ exist with $g'(z_0) \neq 0$ then $\lim\limits_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z)}{g'(z)}$

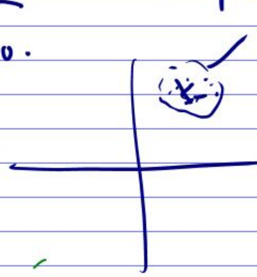
$$\lim \frac{f(z)}{g(z)} = \lim \frac{f(z) - \cancel{f(z_0)}^{\to 0}}{g(z) - \cancel{g(z_0)}^{\to 0}} = \lim_{z \to z_0} \frac{\frac{f(z) - f(z_0)}{z - z_0}}{\frac{g(z) - g(z_0)}{z - z_0}} = \frac{f'(z_0)}{g'(z_0)}$$

Panel 8

Analytic Functions

We want to expand on the idea of $\mathbb{C}$-diffble.

Def. A function $f: D \to \mathbb{C}$ is analytic at a point $z_0 \in D$ if f is $\mathbb{C}$-diffble at z_0 and at all points in a (small) disk around z_0.



Ex: $f(z) = |z|^2$ $\mathbb{C}$-diffble only at $z_0 = 0$
But not analytic

$f(z) = 1/z$ is analytic for all $z \neq 0$

Panel 9

Def. If a function $f: \mathbb{C} \to \mathbb{C}$ is analytic for all $z \in \mathbb{C}$ then f is called entire.

Ex: Any polynomial in z
e^z entire?

$$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x\left(\cos(y) + i\sin(y)\right)$$

are CR: $u(x,y) = e^x\cos(y)$, $v(x,y) = e^x\sin(y)$

$$\left.\begin{array}{ll} u_x = e^x\cos(y) & v_x = e^x\sin(y) \\ u_y = (-)e^x\sin(y) & v_y = e^x\cos(y) \end{array}\right\} \begin{array}{l} f \text{ is } \mathbb{C} \text{ diffble} \\ \text{everywhere} \\ \to \text{entire} \end{array}$$

Panel 10

Theorem: If $f(z)$ and $\overline{f(z)}$ are both analytic, then f must be constant.

Hint: Use CR: $f(z) = u(x,y) + iv(x,y)$ and $\overline{f(z)} = u(x,y) - iv(x,y)$ are analytic

$= u + i(-v)$

$u_x = v_y \qquad u_x = -v_y$

$u_y = -v_x \qquad u_y = +v_x$

$\Rightarrow u_y = -v_x = v_x \Rightarrow u_y = 0 \quad u_x = 0 \to u$ is constant

$\Rightarrow v_x = v_y = 0 \Rightarrow v$ is constant

$f = u + iv$ is constant

Panel 11

Theorem: Suppose $f(z)$ is analytic in a domain D and $|f(z)|$ is constant $\Rightarrow f$ is constant

(Proof) Say $|f(z)| = c \quad \forall z$

1. $c = 0 \quad \Rightarrow f = 0$
2. $c \neq 0 \quad |f(z)|^2 = f \cdot \bar{f} = c^2 \Rightarrow f, \bar{f} \neq 0 \ \forall z$

$\Rightarrow f = c^2 / \bar{f}(z)$ is analytic because $\bar{f}$ is analytic

$\bar{f} = c^2 / f(z)$ is also analytic

$\Rightarrow f$ and $\bar{f}$ are analytic $\Rightarrow f$ constant (see before)

Panel 12

Next we want to investigate the connection between analytic functions $f(z)$ and the component functions

Def: A function $u(x,y)$ is called harmonic if $f_{xx} + f_{yy} = 0$

Note $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ Laplace operator

f is harmonic if $\Delta f = 0$

Panel 13

Ex: Which of these are harmonic functions:

$u(x,y) = x^3 - 3xy^2$

$u_x = 3x^2 - 3y^2 \qquad u_{xx} = 6x$

$u_y = -6xy \qquad u_{yy} = \underline{-6x}$

0 ✓

$v(x,y) = 3x^2y + y^3$

$u_x = 6xy \qquad u_{xx} = 6y$

$u_{y} = 3x^2 + 3y^2 \qquad u_{yy} = 6y$

} not h!

$T(x,y) = e^{-y}\sin(x)$

HW

Panel 14

Theorem: If f is analytic in a domain D and $f(z) = u(x,y) + iv(x,y)$ then both u, v are harmonic!!

Proof: $u_x = v_y \quad u_{xx} = v_{yx}$

$u_y = -v_x \quad u_{yy} = -v_{xy}$

$\Rightarrow u_{xx} + u_{yy} = 0$

Example: Show that $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$ are both harmonic. Explain.

$u_x = 3x^2 - 3y^2, \quad u_{xx} = 6x$

$u_y = -6xy \quad u_{yy} = -6x$ ✓

$\underbrace{u_{xx} + u_{yy}}_{0}$

$v_x = 6xy \quad v_{xx} = 6y$

$v_y = 3x^2 - 3y^2 \quad v_{yy} = \underset{0}{-6y}$

Panel 15

$f = x^3 + iy^2 + x^2y^4$

$= \underbrace{x^6 + x^5y^4}_{} + iy^2$ not harmonic

Panel 16

If $u(x,y) = x^3 - 3xy^2$

$v(x,y) = 3x^2y - y^3$

Then $(x^3 - 3xy^2) + i(3x^2y - y^3) = (x+iy)^3 = z^3$