Panel 1

Summary so far:

Def: $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

Ex: $f(z) = 2z^3 \Rightarrow f'(z) = 6z^2$:)

$f(z) = \bar{z}$ nowhere diff'ble
$f(z) = |z|^2$ diff'ble only at zero! } ???

Q: Since $f(z) = u(x,y) + i\,v(x,y)$, what, if any, relation between $f'(z)$, u_x, u_y, v_x, v_y

Panel 2

$\boxed{f(z) = z^2 - (2+3i)z - (2-4i) = 0}$, $z = 2i$, ? solve w/o WA ~~HW~~

$f(2i) = -4 - (2+3i)(2i) - (2-4i) = (z - 2i)(z + (A+B)i)$
$= -4 - 4i + 6 - 2 + 4i = 0$

$f(2+i) = (2+i)^2 - (2+3i)(2+i) - (2-4i)$
$\cancel{3} + \cancel{4i} \cancel{-4} + \cancel{3} - 2i - \cancel{6i} \cancel{-2} \cancel{+4i} = 0$

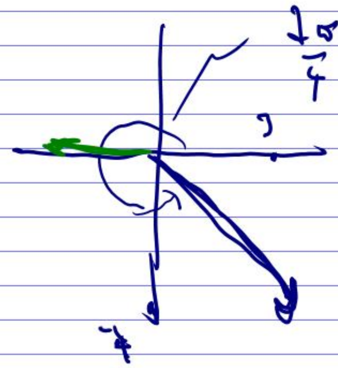
$z^2 - (2+3i)z - (2-4i) = (z - (2+i))(z - 2i)$

$$z = \frac{(2+3i) \pm \sqrt{(2+3i)^2 + 4(2-4i)}}{2} = \frac{2+3i \pm \sqrt{-5+12i+8-16i}}{2}$$

$$= \frac{2+3i \pm \sqrt{3-4i}}{2}$$

Panel 3

$\sqrt{3-4i}$



$\frac{7\pi}{8}$

$\frac{(z-(2+i))(\cancel{z-2i})}{\cancel{(z-2i)}} \Longrightarrow \underline{\underline{0}}$

Could have gotten 0 right away!

Panel 4

$f(z) = x^2 - y^2 + 2ixy \quad = z^2$

$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$

$= \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2\cancel{i}}\right)^2 + \cancel{2i}\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{\cancel{2i}}\right)$

$= \frac{1}{4}\left(\cancel{z^2} + \cancel{2z\bar{z}} + \cancel{\bar{z}^2} + \cancel{z^2} - \cancel{2z\bar{z}} + \cancel{\bar{z}^2} + 2(z^2 - \cancel{\bar{z}^2})\right)$

$= \frac{1}{4}\left(4z^2\right) = \underline{\underline{z^2}}$

Panel 5

Ex: $f(z) = x^2 - y^2 + 2ixy = z^2$

$f'(z) = 2z = 2x + 2iy \quad = u_x + i v_x$

$u_x = 2x \qquad f' = u_x - i u_y$

$u_y = -2y \qquad = v_y + i v_x$

$v_x = 2y$

$v_y = 2x$

Thus: $u_x = v_y$

$u_y = - v_x$

Panel 6

Cauchy - Riemann Equations:

Suppose $f(z)$ is $\mathbb{C}$-diff'ble at z_0. Then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \text{ exists}$$

$$\lim_{(x,y) \to (x_0,y_0)} \frac{u(x,y) + iv(x,y) - u(x_0,y_0) + iv(x_0,y_0)}{x+iy - (x_0+iy_0)} =$$

$$\lim_{(x,y) \to (x_0,y_0)} \frac{u(x,y) - u(x_0,y_0)}{x-x_0 + i(y-y_0)} + i \lim_{(x,y) \to (x_0,y_0)} \frac{v(x,y) - v(x_0,y_0)}{(x-x_0) + i(y-y_0)}$$

Panel 7

$$f'(z_0) = \lim_{(x,y)\to(x_0,y_0)} \frac{u(x,y)-u(x_0,y_0)}{x-x_0+i(y-y_0)} + i\,\frac{v(x,y)-v(x_0,y_0)}{x-x_0+i(y-y_0)}$$

Let $y=y_0$, $x\to x_0$: $$f'(z_0) = \lim_{x\to x_0} \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i\lim_{x\to x_0}\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0} =$$

$$= u_x + i v_x$$

Thm: $f'(z) = u_x + i v_x$

Panel 8

$$f'(z_0) = \lim_{(x,y)\to(x_0,y_0)} \frac{u(x,y)-u(x_0,y_0)}{x-x_0+i(y-y_0)} + i\,\frac{v(x,y)-v(x_0,y_0)}{x-x_0+i(y-y_0)}$$

$x=x_0$, $y\to y_0$: $$f'(z_0) = \lim_{y\to y_0} \frac{u(x_0,y)-u(x_0,y_0)}{i(y-y_0)} + \cancel{i}\lim_{y\to y_0}\frac{v(x_0,y)-v(x_0,y_0)}{\cancel{i}(y-y_0)}$$

$$= \frac{1}{i}u_y + v_y$$

Thm 2 $f'(z) = -i\,u_y + v_y$

Panel 9

We proved that: if $f(z)$ is $\mathbb{C}$-diff'ble, then

$$f'(z) = u_x + iv_x = -iu_y + v_y$$

Theorem: If f is $\mathbb{C}$-diff'ble, and $f(z) = u(x,y) + iv(x,y)$, then

$$u_x = v_y \qquad \text{Cauchy-Riemann}$$
$$u_y = -v_x \qquad \text{Equations (CR)}$$

Also: $f'(z) = u_x + iv_x = v_y - iu_y$

Panel 10

Ex: Suppose $f(z) = z^3$. Show that CR hold.

$$= (x+iy)^3 = (x+iy)^2(x+iy)$$
$$= (x^2 - y^2 + 2ixy)(x+iy) =$$
$$= x^3 - xy^2 + 2ix^2y + ix^2y - iy^3$$

$$= x^3 - xy^2 \overset{-2xy^2}{-2xy^2} + i(2x^2y + x^2y - y^3) =$$
$$= \underbrace{x^3 - 3xy^2}_{u} + i\underbrace{(3x^2y - y^3)}_{v} =$$

Panel 11

$$\underbrace{x^3 - 3xy^2}_{u} + i\left(\underbrace{3x^2y - y^3}_{v}\right).$$

$$u_x = 3x^2 - 3y^2 = v_y = 3x^2 - 3y^2$$

$$u_y = -6xy = -v_x = 6xy$$

$\boxed{\text{CR check out}}$

Panel 12

Is $f(z) = \underbrace{x^2+y^2}_{u} + \underbrace{2ixy}_{v}$ $\mathbb{C}$-diffble

(1) Replace $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$. Simplify. All $\bar{z}$'s disappear $\Leftrightarrow$ $\mathbb{C}$-diffable

CR: $u_x = 2x = v_y = 2x$

$u_y = 2y \neq -v_x = 2y$

CR don't check out!

$\Rightarrow$ Not $\mathbb{C}$-diffble

Panel 13

$f(z) = i x^5 + x^3 y^2 + i x^4 y^1 + x^2 y^5$

Is f $\mathbb{C}$-diffble ??? No!

$u(x,y) = x^3 y^2 + x^2 y^5 \qquad v(x,y) = x^5 + x^4 y$

$u_x = 3x^2 y^2 + 2xy^5 \neq v_y = x^4$

Replace $x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$: there will be $\bar{z}$'s

left after simplifying!