Panel 1

Functions $f: \mathbb{C} \to \mathbb{C}$

$$f(z) = z^2 + i = (x+iy)^2 + i = x^2 + 2ixy - y^2 + i$$

$$f = u(x,y) + i\,v(x,y) \qquad = \underbrace{x^2 - y^2}_{u(x,y)} + i\,\underbrace{(2xy+1)}_{v(x,y)}$$

Also, could use $z = x + iy$, $\bar{z} = x - iy$, $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$$\Rightarrow H(x,y) = f(z,\bar{z}) + i\,g(z,\bar{z})$$

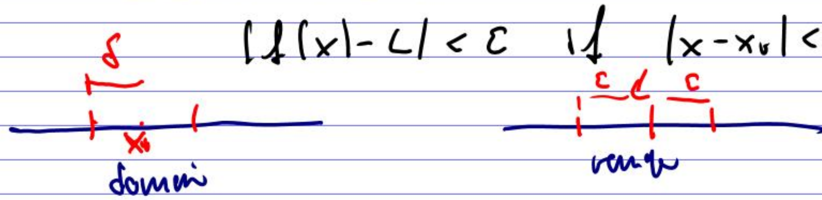
Panel 2

Limits and Continuity: $f: \mathbb{R} \to \mathbb{R}$

Recall: $\lim_{x \to x_0} f(x) = L$ means: take $\varepsilon > 0$

there is $\delta > 0$ s.t.

$$|f(x) - L| < \varepsilon \quad \text{if} \quad |x - x_0| < \delta$$



unit circle $|z| = 1$

$f: \mathbb{C} \to \mathbb{C}$ $\quad \lim_{z \to z_0} f(z) = L \quad$ iff given $\varepsilon > 0$

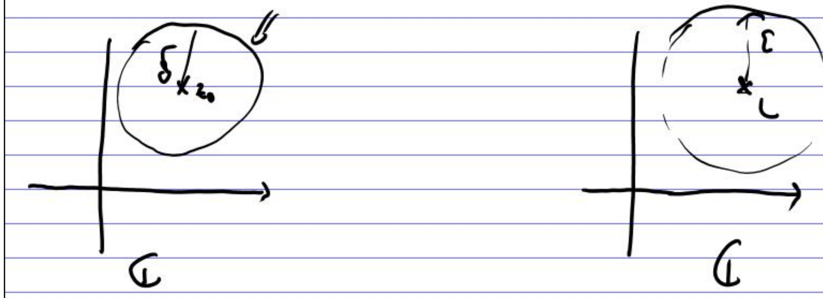
there is $\delta > 0$ s.t. $\qquad$ Recall: $|z - z_0| < \delta$

$$|f(z) - L| < \varepsilon \quad \text{if} \quad |z - z_0| < \delta$$

Panel 3

The Trouble with Limits in $\mathbb{C}$: $\lim_{z\to z_0} f(z) = L$ means

$|z-z_0|<\delta \Rightarrow |f(z)-L|<\varepsilon$



Note: $|z|=1 \Leftrightarrow \sqrt{x^2+y^2}=1 \Leftrightarrow x^2+y^2=1$ unit circle

$|z-z_0|<\delta \Leftrightarrow |x-x_0+i(y-y_0)|<\delta$

$\Leftrightarrow \sqrt{(x-x_0)^2+(y-y_0)^2}<\delta \Rightarrow (x-x_0)^2+(y-y_0)^2<\delta^2$

Panel 4

For complex limits we cannot resort to left/right limits, because there are inf. many paths to approach z_0

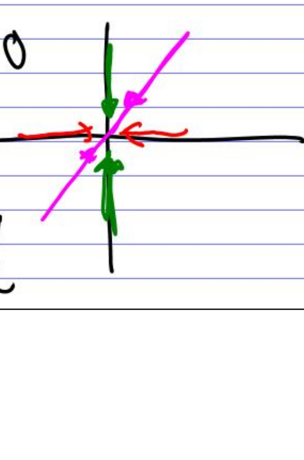
Thus: Same as $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

try to find approach paths with different answers.

$y=0,\ x\to 0$: $\lim_{x\to 0} \frac{0}{x^2}=0$

$x=0,\ y\to 0$: $\lim \frac{0}{y^2}=0$

$x=y\to 0$: $\lim \frac{x^2}{2x^2}=\frac{1}{2}$



Limits are different, so lim d.n.e.

Panel 5

Ex: $f(z) = z/\bar{z}$. Then $\lim_{z \to 0} f(z)$ does not exist:

$$f(z) = \frac{z}{\bar{z}} = \frac{x+iy}{x-iy}$$

1) Take $y=0$: $\lim_{x \to 0} \frac{x+i0}{x-i0} = 1$

2) Take $x=0$: $\lim_{y \to 0} \frac{iy}{-iy} = -1$

So $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist!

Panel 6

Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be a complex function defined in a nbhd. ~ neighbourhood $z_0 = (x_0, y_0)$. Then

$$\lim_{z \to z_0} f(z_0) = w_0 = u_0 + i v_0$$

iff $\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u(x_0,y_0) = u_0$

and $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v(x_0,y_0) = v_0$

Thus: limits (and continuity) of complex functions are exactly the same as $\lim_{(x,y) \to (x_0,y_0)} u(x,y)$. Boooring!!

Panel 7

Def: f a complex function defined in a nbhd of z_0. Then f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Thm: $f = u + iv$ is cont. in $\mathbb{C}$ iff both u, v are cont. in $\mathbb{R}^2$

Ex: $\lim_{x \to 0} 2x^2 - 5 = -5$

Panel 8

Continuity Theorems

- A polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ is cont.
- A rational function $r(z) = p(z)/q(z)$ is cont. as long as $q(z) \neq 0$

$\frac{1}{z^2 - 1}$

- The sum of two continuous functions is cont.
- The product of two continuous functions is cont.
- The quotient of two continuous functions is cont. except where denom equals zero!

Panel 9

Ex: Find $\lim_{z\to 1+i} \frac{z^2-2i}{z^2-2z+2} = \frac{(1+i)^2-2i}{(1+i)^2-2(1+i)+2} = \frac{0}{0}$

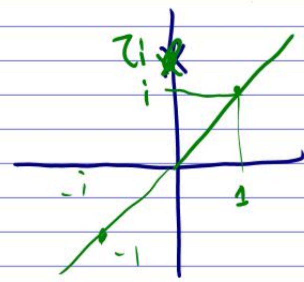
$$\frac{z^2-2i = (\cancel{z-(1+i)})(z+(1+i))}{0 = z^2-2z+2 = (\cancel{z-(1+i)})(z-(1-i))} = \frac{2+2i}{2i}$$

long division

quadratic: $0 = z^2-2z+2 \Rightarrow z = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm 2i}{2}$

$z^2-2i=0 \qquad z^2=2i$

$z=\sqrt{2i}$



Panel 10

Derivatives For $f(z)$ a complex function defined in a nbhd of z_0 we define

$$f'(z) \overset{(1)}{=} \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} \overset{(2)}{=} \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

If limit exists, f is called $\mathbb{C}$-diffble

Ex $f(z)=z^3 \Rightarrow f'(z)=3z^2$

Proof $z^3-z_0^3 = (z-z_0)(z^2+zz_0+z_0^2)$

Thus $\lim \frac{z^3-z_0^3}{z-z_0} = \lim_{z\to z_0}\frac{\cancel{(z-z_0)}(z^2+zz_0+z_0^2)}{\cancel{(z-z_0)}} = 3z_0^2$

Panel 11

Ex: Show that $f(z) = \bar{z}$ is nowhere diff'ble

(New concept!)

$z_0 = x_0 + iy_0$

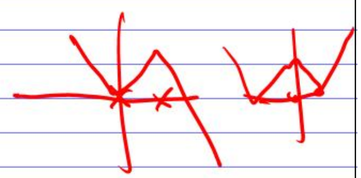
$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{(x-iy)-(x_0-iy_0)}{x-x_0+i(y-y_0)} =$$

$$= \lim_{(x,y)\to(x_0,y_0)} \frac{(x-x_0)-i(y-y_0)}{(x-x_0)+i(y-y_0)}$$

D.N.E. for any z!

check $x = x_0$: $\lim \frac{-i(y-y_0)}{i(y-y_0)} = -1$

$y = y_0$: $\lim \frac{x-x_0}{x-x_0} = 1$



Panel 12

Ex: Let $f(z) = |z|^2$. Is it differentiable? Where?

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{|z|^2-|z_0|^2}{z-z_0} =$$

$$= \lim_{(x,y)\to(x_0,y_0)} \frac{(x^2+y^2)-(x_0^2+y_0^2)}{x-x_0+i(y-y_0)}$$

$x = x_0$: $\lim \frac{y^2-y_0^2}{i(y-y_0)} = \lim \frac{(\cancel{y-y_0})(y+y_0)}{i(\cancel{y-y_0})} = \frac{2y_0}{i} = -2iy_0$

$y = y_0$: $\lim_{x\to x_0} \frac{x^2-x_0^2}{x-x_0} = 2x_0$

$\Rightarrow 2x_0 = -2iy_0 \Rightarrow x_0 = y_0 = 0$

Panel 13

Summary so far.

$f(z) = z^3 \Rightarrow f'(z) = 3z^2$, nice!

$f(z) = \bar{z} \rightarrow$ not diff'ble anywhere ??

$f(z) = |z|^2 \Rightarrow$ diff'ble only at 0 !!!! ????!!!

Happy :-) because I found something new!

$\mathbb{C}$-diff'ble will be like $\mathbb{R}$-diff'ble and then some!!

Next!