Panel 1

Last Time: Root Day

$z^n = \boxed{a = re^{i\theta}}$ always has n different solutions

Algebraically: $z_k = r^{1/n} e^{i(\theta/n + \frac{2k\pi}{n})}$, $k = 0, 1, \dots, n-1$

Geometrically: roots are symmetrically distributed and form a regular polygon with n sides starting at the principle angle θ/n

Roots of Unity: $z^n = 1 = 1 \cdot e^{i0}$

$z_k = e^{i\frac{2k\pi}{n}}$, $k = 0, 1, \dots, n-1$

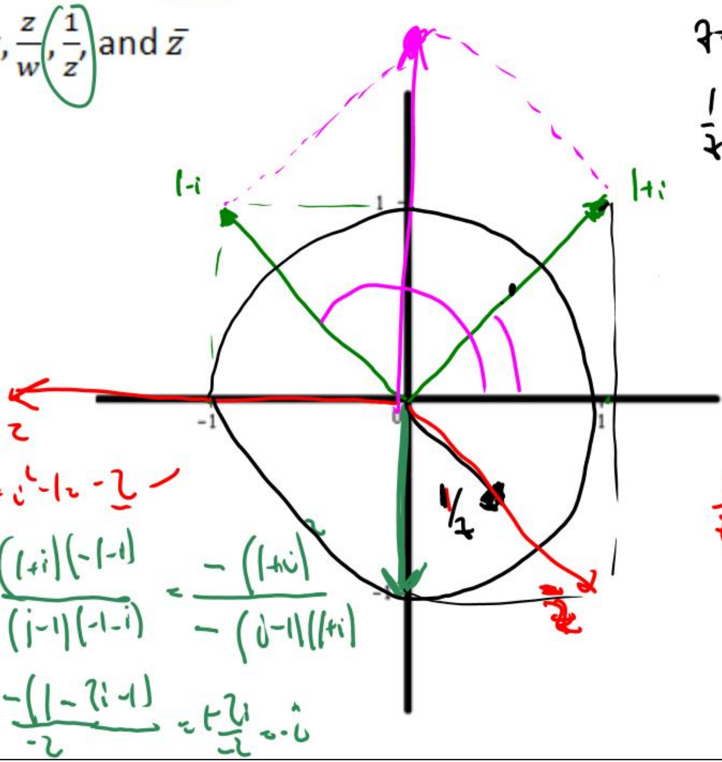
Panel 2

Consider $z = 1 + i$ and $w = -1 + i$. Draw the vectors $z, w, z \cdot w, \frac{z}{w}, \frac{1}{z}$, and $\bar{z}$

$z + w$

$z = re^{i\theta}$

$\frac{1}{z} = \frac{1}{r} e^{-i\theta}$



$(i+1)(i-1) = i^2 - 1 = -2$

$\frac{z}{w} = \frac{1+i}{i-1} = \frac{(1+i)(-1-i)}{(i-1)(-1-i)} = \frac{-(1+i)^2}{-(i-1)(1+i)}$

$= \frac{-(1-2i-1)}{-2} = \frac{+2i}{-2} = -i$

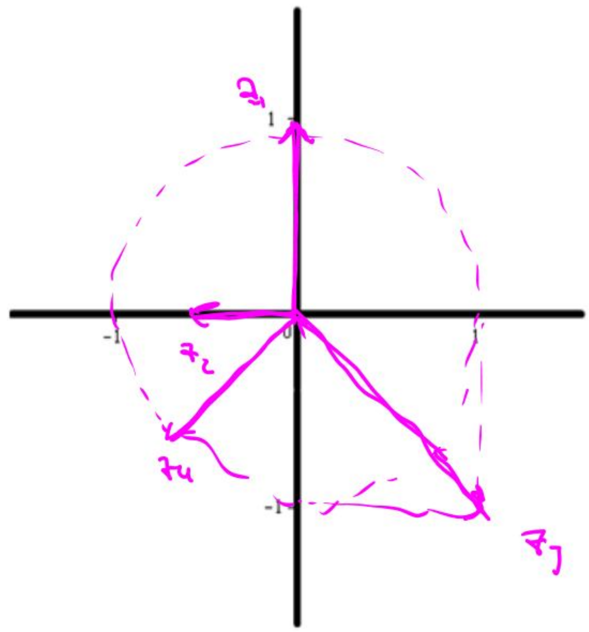
$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$

Panel 3

Draw the following vectors: $z_1 = e^{\frac{i\pi}{2}}$, $z_2 = 0.5e^{i\pi}$, $z_3 = \sqrt{2}e^{\frac{-i\pi}{4}}$,

$= i$ $= -0.5$

and $z_4 = e^{\frac{i5\pi}{4}}$



Panel 4

Describe in simple geometric terms what happens to a vector z when:

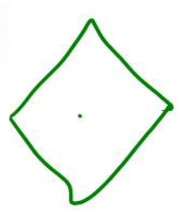
a. it is multiplied by 2

length doubles!

b. it is multiplied by -1

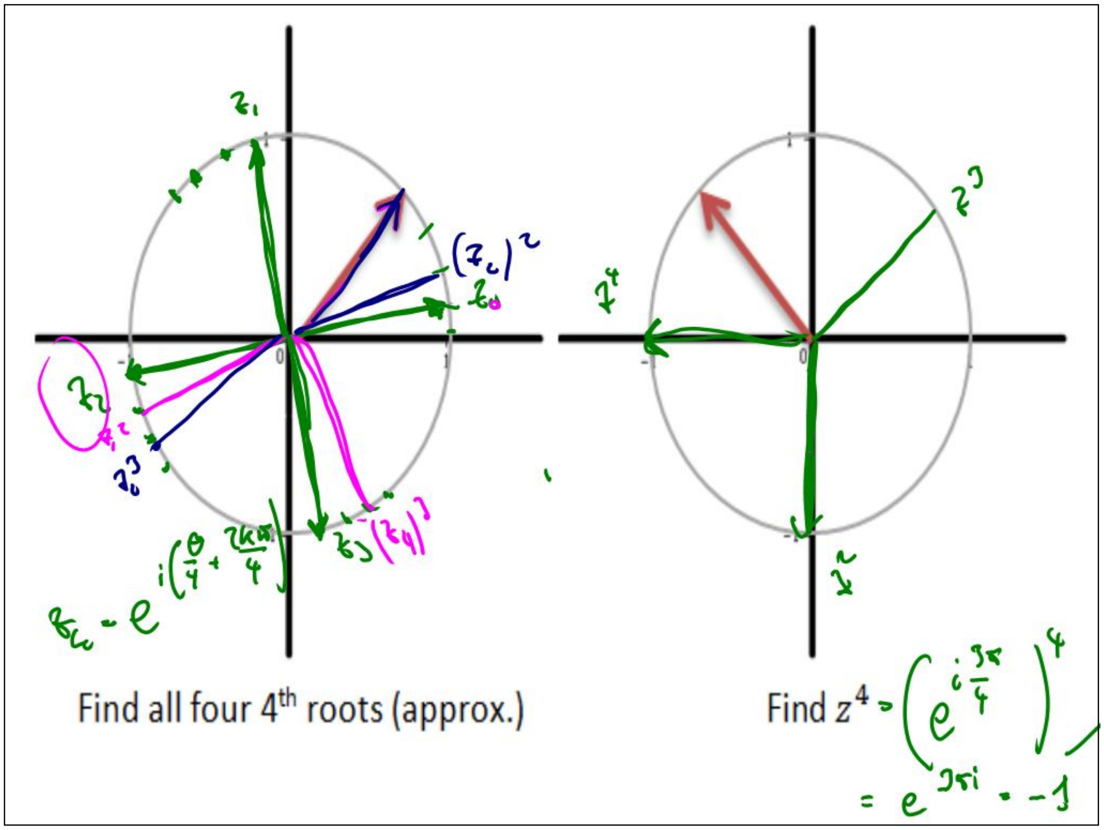
flips

c. it is multiplied by i rotation by $\pi/2$ (90°)

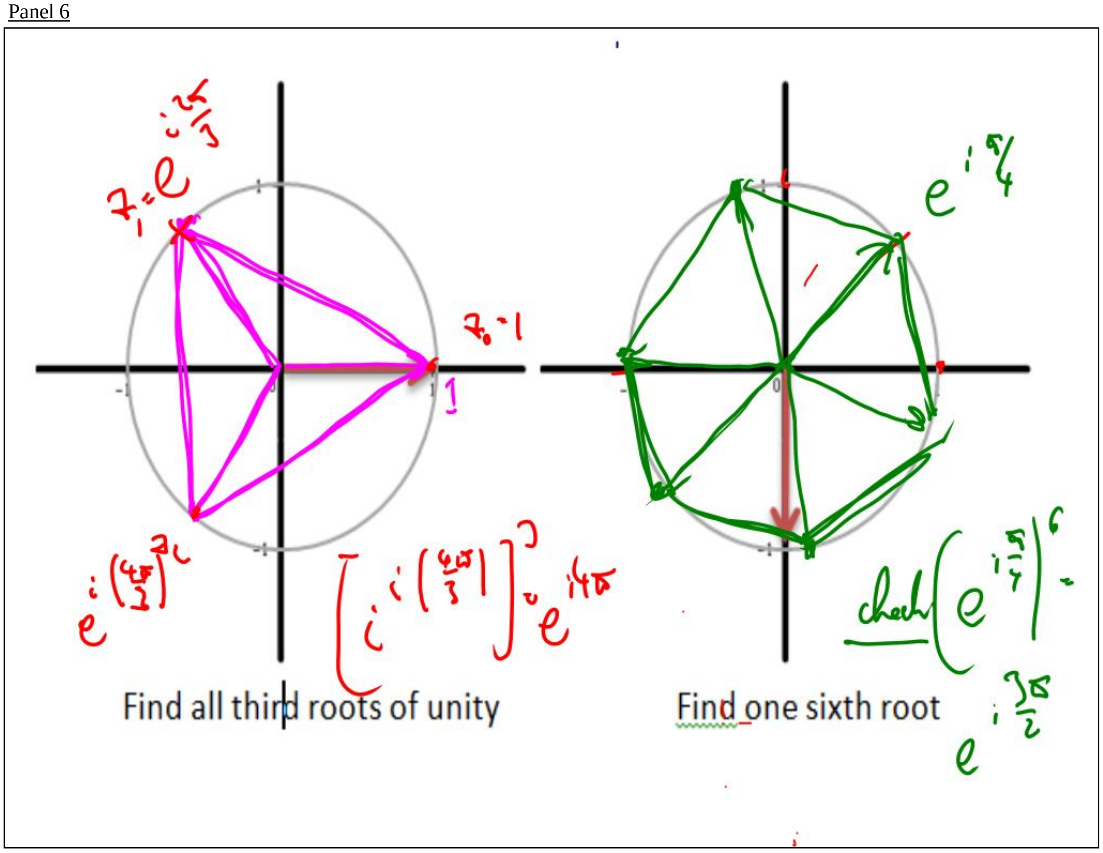


d. it is squared angle doubles

Panel 5

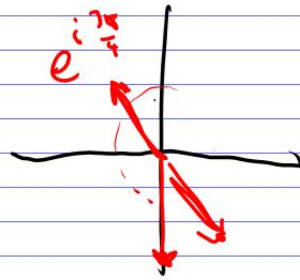


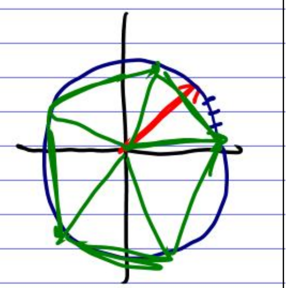
Panel 6



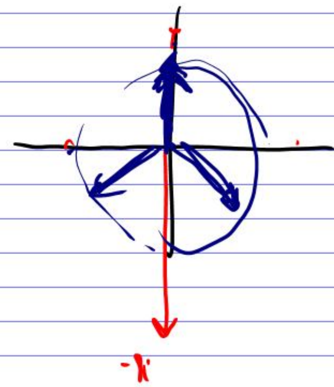
Panel 7

$\sqrt{-i} =$





$(-8i) =$



Panel 8

$\omega_k = e^{i\frac{2k\pi}{n}}, \quad k = 0, 1, \dots, n-1$ are n-th roots of 1

i.e. $(\omega_k)^n = \left(e^{i\frac{2k\pi}{n}}\right)^n = e^{i2k\pi} = 1$

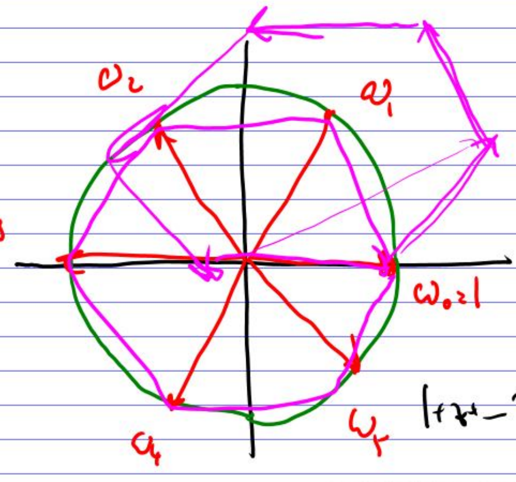
$1 + \omega_1 + \omega_2 + \dots + \omega_{n-1} =$

$1 + e^{i\frac{2\pi}{n}} + e^{i\frac{4\pi}{n}} + e^{i\frac{6\pi}{n}} + \dots + e^{i\frac{(n-1)2\pi}{n}}$

$1 + \left(e^{i\frac{2\pi}{n}}\right) + \left(e^{i\frac{2\pi}{n}}\right)^2 + \left(e^{i\frac{2\pi}{n}}\right)^3 + \dots + \left(e^{i\frac{2\pi}{n}}\right)^{n-1} = \frac{1-z^n}{1-z}$

$z = e^{i\frac{2\pi}{n}}: \quad \frac{1-\left(e^{i\frac{2\pi}{n}}\right)^n}{1-e^{i\frac{2\pi}{n}}} = \frac{0}{1-e^{i\frac{2\pi}{n}}} = 0$ ✓

Panel 9



$n=6$

$\omega_k = e^{i\frac{2k\pi}{6}}$

$1+z+\dots+z^{n-1} = S = \frac{1-z^n}{1-z}$

$$S = 1 + \cancel{z} + \cancel{z^2} + \cancel{z^3} + \cancel{\dots} + \cancel{z^{n-1}}$$

$$zS = \cancel{z} + \cancel{z^2} + \cancel{z^3} + \cancel{z^4} \dots + z^n$$

$$= S - zS = 1 - z^n \quad \Rightarrow S(1-z) = 1 - z^n$$

Panel 10

Complex Functions

A complex function is a rule that assigns to every z in a domain $D \subset \mathbb{C}$ a complex number w.

Ex: $f(z) = z^2$ $\qquad g(z) = x^2 + y^2$

$k(z) = \frac{5+z}{z^2-1}$ $\qquad h(z) = z - \bar{z}$

Note: If no domain is specified explicitly, we assume the largest possible subset of $\mathbb{C}$ as domain.

Panel 11

Thm: Every complex function $f(z) = w$ can be written as $f(z) = u(x,y) + iv(x,y)$

Ex: $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + \underbrace{2ixy}_{v(x,y) = 2xy}$

$g(z) = z\bar{z} = \|z\|^2 = x^2 + y^2$ $\quad u(x,y) = x^2 + y^2$

$v(x,y) = 0$

$h(z) = ix^2$ $\quad u(x,y) = 0$

$v(x,y) = x^2$

Panel 12

Thm: Every function $u(x,y) + iv(x,y)$ can be converted to a function $f(z, \bar{z})$

Recall: $x = \frac{z+\bar{z}}{2}$ $\qquad y = \frac{z-\bar{z}}{2i}$

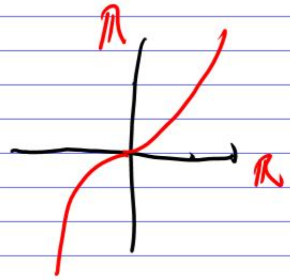
Ex: $u(x,y) + iv(x,y) = x^2 + iy^2 = f(z)$

$f(z) = \left(\frac{z+\bar{z}}{2}\right)^2 + i\left(\frac{z-\bar{z}}{2i}\right)^2 = f(z,\bar{z})$

Panel 13

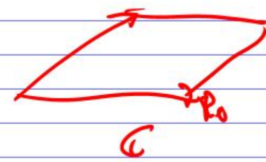
Graphs of Complex Functions

$f:\mathbb{R}\to\mathbb{R}$



graph in 2D of $f:\mathbb{R}\to\mathbb{R}$

$f:\mathbb{C}\to\mathbb{C}$



graph of $f:\mathbb{C}\to\mathbb{C}$

is 4D !!!

Panel 14

Graphing Complex Functions

Next Monday!