Panel 1

Last time:

$2e^{i\pi} = -2$

$e^{-i\pi/4} = \frac{1}{\sqrt{2}}(1-i)$

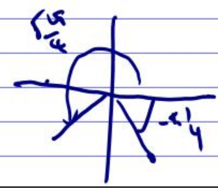
$z = x+iy = re^{i\theta}$

$z+w$: vector addition in $\mathbb{R}^2$

$z \cdot w$ = mult. lengths, add angles!

z/w = divide lengths, subtract angles. Ex: $\frac{1}{\sqrt{2}}\underbrace{(1-i)}_{\sqrt{2}} \cdot \underbrace{(-1-i)}_{\sqrt{2}} = \sqrt{2}e^{i\pi} = -\sqrt{2}$

$z^n = \underset{\sim}{r^n}$, $\underline{n \cdot \text{angle}}$



Panel 2

Important Theorems

Euler's Theorem: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Euler's Formula: $e^{i\pi} + 1 = 0$

De Moivre Theorem: $\left(e^{i\theta}\right)^n = e^{in\theta}$

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

Panel 3

$z = e^{i\frac{3\pi}{4}} \quad w = e^{i\frac{3\pi}{4}}$

$\arg(z) = \frac{3\pi}{4}, \quad \arg(z \cdot w) = \arg\left(e^{i\frac{6\pi}{4}}\right) = \frac{6\pi}{4} = \underbrace{\arg(z)}_{\frac{3\pi}{4}} + \underbrace{\arg(w)}_{\frac{3\pi}{4}}$

$\operatorname{Arg}(zw) \neq \operatorname{Arg}(z) + \operatorname{Arg}(w)$

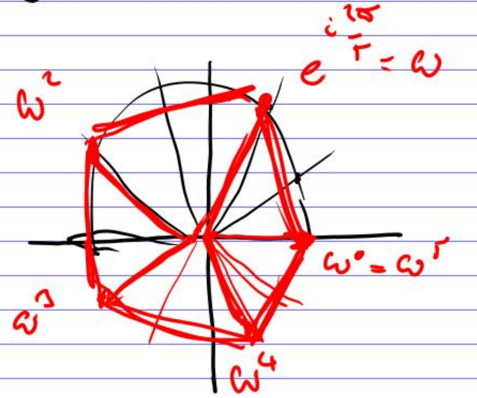
$-\frac{\pi}{2} = \operatorname{Arg}\left(e^{i\frac{6\pi}{4}}\right) \neq \frac{3\pi}{4} + \frac{3\pi}{4}$

Panel 4

$(\cos(\theta) + i\sin(\theta))^3 = \cos(3\theta) + i\sin(3\theta)$

$\underbrace{[\cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)]}\ + \underbrace{[3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)]\, i}$

$\omega = e^{i\frac{2\pi}{5}}$

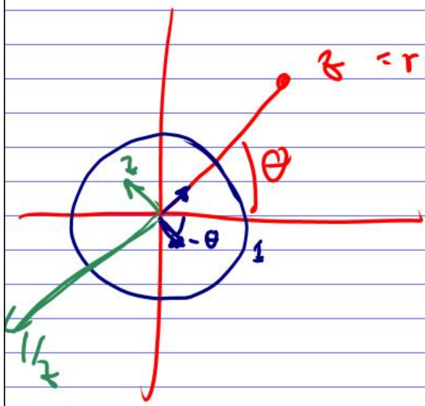


$\omega^0 = 1$

$\omega^5 = \left(e^{i\frac{2\pi}{5}}\right)^5 = e^{i\frac{2\pi}{5}\cdot 5}$

$= e^{i2\pi} = 1$

Panel 5



$$z = re^{i\theta} \Rightarrow \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

Panel 6

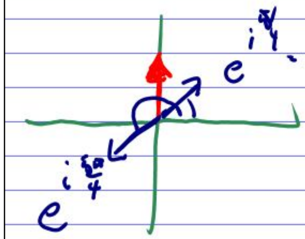
Complex Roots

Find geometrically $\sqrt{i}$, $\sqrt[4]{1}$, $\sqrt[3]{-1}$

2 answers

4 answers $(1, -, -, -)$

-1, 2 more



$$e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)$$

$$e^{i\frac{5\pi}{4}}$$

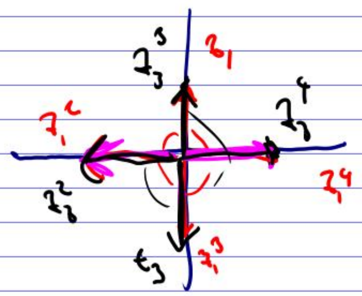
$$\frac{1}{\sqrt{2}}(-1-i)$$

$$e^{-i\frac{3\pi}{4}}$$

$$\left(e^{-i\frac{3\pi}{4}}\right)^2 = e^{-i\frac{3\pi}{4}\cdot 2} \quad e^{-i\frac{3\pi}{2}} = i$$

Panel 7

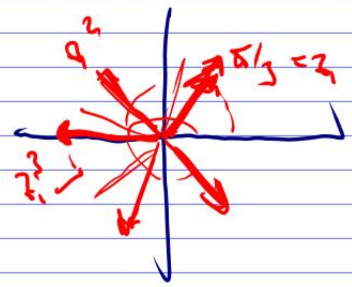
$\sqrt[4]{1}$



1st angle: $\theta/4$

$1, i, -1, -i$

$\sqrt[3]{-1}$



$z_1 = e^{i\pi/3}$

$z_2 = e^{i\pi}$ $\quad (z_2)^3 = e^{3\pi i} = -1$

$z_3 = e^{i\frac{5\pi}{3}}, \quad (z_3)^3 = e^{i5\pi} = -1$

Panel 8

To find n-th root of a: $\quad$ solve $z^n = a = re^{i\theta}$

$$z = (re^{i\theta})^{1/n} = r^{1/n}(e^{i\theta})^{1/n} = r^{1/n}e^{i\frac{\theta}{n}} =$$

$$= r^{1/n}e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, \dots n-1$$

Proof

$$\left[r^{1/n}e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)}\right]^n = re^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)n}$$

$$= re^{i(\theta + 2k\pi)} =$$

$$= re^{i\theta} = a$$

Panel 9

Ex: Find all 4^{th} roots of i

Thm: $z^n = a = re^{i\theta} \Rightarrow z = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}$, $k = 0, 1, 2, \ldots, n-1$

Solve $z^4 = i = e^{i\frac{\pi}{2}} \Rightarrow z_0 = e^{i\frac{\pi/2}{4}} = e^{i\pi/8}$

$z_1 = e^{i\left(\frac{\pi/2}{4} + \frac{2\pi}{4}\right)} = e^{i\left(\frac{5\pi}{8}\right)}$

$z_2 = e^{i\left(\frac{\pi/2}{4} + \frac{4\pi}{4}\right)} = e^{i\frac{9\pi}{8}}$

$z_3 = e^{i\left(\frac{\pi/2}{4} + \frac{6\pi}{4}\right)} = e^{i\frac{13\pi}{8}}$

$z_4 = e^{i\left(\frac{\pi/2}{4} + \frac{8\pi}{4}\right)} = e^{i\pi/8}$

Panel 10

$\sqrt[6]{i}$:

$z^n = a = re^{i\theta}$

$z_k = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}$, $k = 0, 1, \ldots, n-1$

$z^6 = i = e^{i\pi/2}$

$n = 6, \ \theta = \frac{\pi}{2}$

$z_0 = e^{i\pi/12}$

$z_1 = e^{i\left(\frac{\pi}{12} + \frac{2\pi}{6}\right)} = e^{i\left(\frac{5}{12}\pi\right)}$

$z_2 = e^{i\left(\frac{\pi}{12} + \frac{4\pi}{6}\right)} = e^{i\left(\frac{9\pi}{12}\right)}$

$z_3 = e^{i\frac{13\pi}{12}}$

$z_4 = e^{i\frac{17\pi}{12}}$

$z_5 = e^{i\frac{21\pi}{12}}$

$z_6 = e^{i\frac{25}{12}\pi} = z_0 = e^{i\pi/12}$

$z_7 =$

Test

$\left(e^{i\frac{17\pi}{12}}\right)^6 = e^{i\frac{17}{2}\pi} = e^{i\left(8\pi + \frac{\pi}{2}\right)} = e^{i\pi/2} = i$

Panel 11

<u>Theorem:</u> $z^n = a = re^{i\theta}$ has n different solutions

$$z_i = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, 2, \ldots n-1$$

<u>Ex:</u> Find $\sqrt[2]{1}$ and $\sqrt[3]{-1}$

HW

Panel 12

<u>Theorem:</u> $z^n = a = re^{i\theta}$ has n different solutions

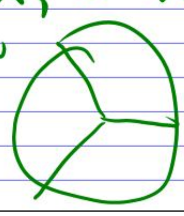
$$z_i = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, 2, \ldots n-1$$

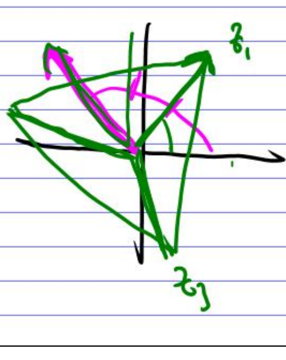
<u>Graphically:</u> 1) Find first root by dividing angle by n

~~2.) Divide 2π into n parts,~~ $\sqrt[3]{-1+i}$

~~draw z~~

2. Divide 2π into n parts starting at first angle





Panel 13

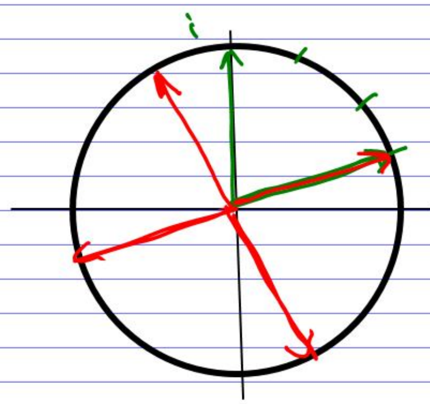
Find $\sqrt[4]{i}$ or $z^4 = i$ graphically

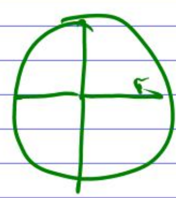
(1) Draw unit circle including i

(2) Divide angle by 4

(3) Draw regular polygon
with 4 corners

(4) Adjust the radius





Panel 14