Panel 1

Last time:

$z \in \mathbb{C}$ as vectors

$\|z\| = \sqrt{x^2+y^2} = \|re^{i\theta}\|$

$\bar{z} = x - iy = re^{-i\theta}$

arg(z) is angle θ in $re^{i\theta}$

Arg(z) is angle θ, $-\pi < \theta \leq \pi$

$z = x + iy$ ✓

$z = r(\cos\theta + i\sin\theta)$

$z = re^{i\theta}$ ✓

$= r\|e^{i\theta}\| = r\|\cos\theta + i\sin\theta\|$

$= r\sqrt{\cos^2\theta + \sin^2\theta} = r$

Note: $e^{i\theta}$ is a circle!

Euler's Theorem: $\cos(\theta) + i\sin(\theta) = e^{i\theta}$

Euler's Formula: $e^{i\pi} + 1 = 0$

Note: $e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$

$= \bar{z}$

$z \cdot w = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 \cdot r_2 e^{i(\theta_1+\theta_2)}$

To multiply z, w:

Panel 2

$z = 3 + 4i$ Find $\sqrt{z} = \sqrt{3+4i} = x + iy$

$(3+4i) = (x+iy)^2$

$x^2 - y^2 + 2xyi = 3 + 4i$

$x^2 - y^2 = 3$ $\quad 2xy = 4$

$\frac{4}{y^2} - y^2 = 3$ $\quad x = \frac{2}{y}$

$4 - y^4 = 3y^2 \Rightarrow -y^4 - 3y^2 + 4 = 0$

$(-y^2 - 4)(y^2 - 1) = 0$ $\quad y^2 = 4$ or 1

$y = \pm 1 \Rightarrow x = \pm 2$ $\quad \pm(2+i)$

Panel 3

$$3+4i = 5e^{i\theta}, \quad \theta = \arctan\left(\frac{4}{3}\right) = 0.9272$$

$$\Rightarrow \sqrt{3+4i} = \left(5e^{i\theta}\right)^{1/2} = 5^{1/2} e^{i\frac{0.9272}{2}}, \quad ?$$

$$|z+i| = 3 \iff \|x+iy+i\| = \|\underbrace{x+i(y+1)}\|$$

$$= \sqrt{x^2+(y+1)^2} = 3$$

$$x^2+(y+1)^2 = 9$$

circle, centered at $(0,-1)$, radius 3

Panel 4

$$\|z-z_0\| = R \iff \|x+iy-(x_0+iy_0)\| = R$$

$$\|x-x_0+i(y-y_0)\| = R$$

$$(x-x_0)^2+(y-y_0)^2 = R^2 \quad \text{circle!}$$

Fact: $z\cdot\bar{z} = (x+iy)(x-iy) = x^2+y^2 = \|z\|^2$

$$\|z-z_0\| = R \iff \|z-z_0\|^2 = R^2$$

Fact: $z+\bar{z} = 2x = 2\,\mathrm{Re}(z)$

$$(z-z_0)\left(\overline{z-z_0}\right) = R^2$$

$$(z-z_0)(\bar{z}-\bar{z}_0) = R^2$$

$$z\bar{z} - (z_0\bar{z} + z\bar{z}_0) + z_0\bar{z}_0 = R^2$$

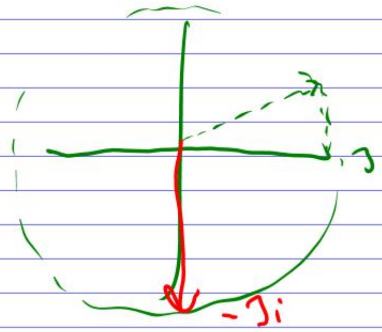
$$\|z\|^2 - \left(\overline{z\bar{z}_0} + z\bar{z}_0\right) + \|z_0\|^2 = R^2$$

$$\|z\|^2 - 2\,\mathrm{Re}(z\bar{z}_0) + \|z_0\|^2 = R^2$$

Panel 5

$$3e^{i\frac{3\pi}{2}} = 3\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right) = 3(0-i) = -3i$$

$3e^{i\frac{3\pi}{2}}$



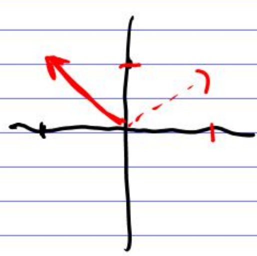
Panel 6

$$(-1+i)^7 = \left(\sqrt{2}\, e^{i\frac{3\pi}{4}}\right)^7 = (\sqrt{2})^7\left(e^{i\frac{3\pi}{4}}\right)^7 =$$

$$= (\sqrt{2})^6\sqrt{2}\left(e^{i\frac{3\pi}{4}\cdot 7}\right) =$$

$$2^3 \cdot 2^{\frac{1}{2}}\, e^{i\frac{21\pi}{4}} =$$

$$8\sqrt{2}\, e^{i\left(\frac{20\pi}{4} + \frac{1\pi}{4}\right)} =$$



$(-1+i)\cdot(-1+i)(-1+i)\ldots(-1+i)$

$$8\sqrt{2}\, e^{i5\pi}\, e^{i\frac{\pi}{4}} = -8\sqrt{2}\, e^{i\pi/4}$$

$$= -8(1+i)$$

Panel 7

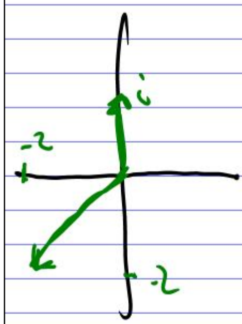
$$z = \frac{i}{-2-2i} = \frac{e^{i\pi/2}}{\sqrt{8}\, e^{i\frac{5\pi}{4}}} = \frac{1}{\sqrt{8}} e^{i\left(\frac{\pi}{2} - \frac{5\pi}{4}\right)} =$$

Recall: $z \cdot w$: mult. radii, add angles

$$z/w = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i\theta_1} e^{-i\theta_2}$$

$$= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

divide radii, subtract angles



$$= \frac{1}{\sqrt{8}} e^{-i\frac{3\pi}{4}} \Rightarrow \arg(z) = -\frac{3\pi}{4}$$

$$\operatorname{Arg}(z) = -\frac{3\pi}{4}$$

Panel 8

Note: $(e^{i\theta})^n = e^{i(n\theta)}$ ✓ De Moivre's Formula

$$\left(\cos(\theta) + i\sin(\theta)\right)^n = \cos(n\theta) + i\sin(n\theta)$$

n=2: $\left(\cos(\theta) + i\sin(\theta)\right)^2 = \cos^2\theta - \sin^2\theta + i\, 2\cos\theta\sin\theta$

$$= \cos(2\theta) + i\sin(2\theta)$$

$$\Rightarrow \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\sin(2\theta) = 2\cos(\theta)\sin(\theta)$$

Panel 9

Visualize operations with complex numbers

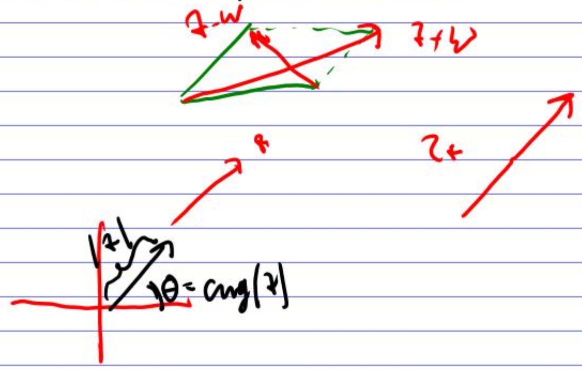
$z + w$

$z - w$

$c \cdot z$, $c \in \mathbb{R}$

$|z|$

$\arg(z)$

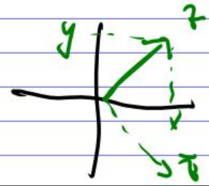


$z \cdot w$ mult lengths, add angles.

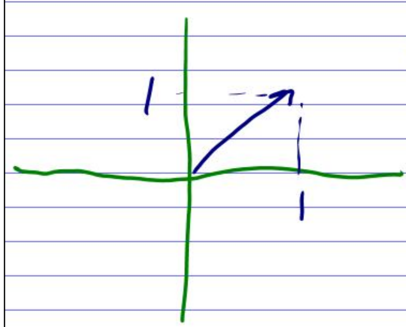
z / w divide lengths, subtract angles

$\bar{z}$ ($\bar{z}$ reflects about x-axis)

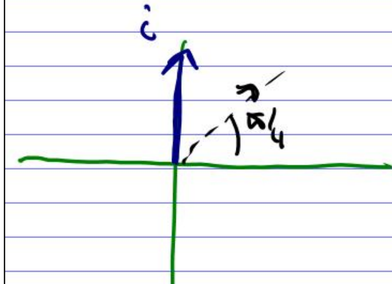
z^n z^n is r^n, n-times angle



Panel 10



$z = 1 + i \Rightarrow z^4 = -4$



$\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$