Panel 1

Last Time:

$z \in \mathbb{C}$ means $z = (x, y) = x + iy, \quad x, y \in \mathbb{R}$

$i = (0, 1)$ s.t. $i^2 = -1$

$z + w$ ✓

$z \cdot w$ ✓

$\mathrm{Re}(z) = x$

$\mathrm{Im}(z) = y$

$z/w = \frac{(x + iy)}{x_2 + iy_2} \frac{(x_2 - iy_2)}{(x_2 - iy_2)} = (A + iB) \frac{1}{x^2 + y^2}$

$z^2 - 2z + 2 = 0$ ✓ $z = 1 + i$ ✓ try $-(1+i)$?

$-1+1$ ✓

$1-1$ ✓

Panel 2

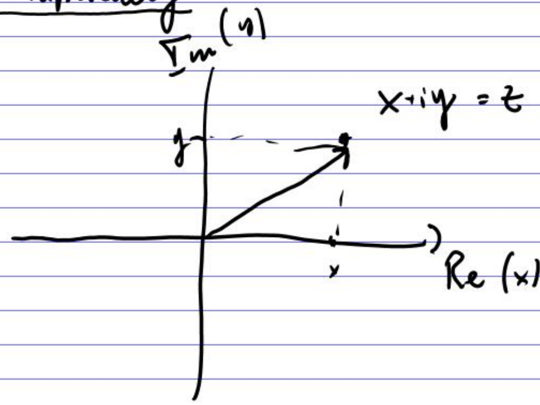
$z^2 + 2z - 2 = (z - (1+i))(z - w) = 0$

$$\begin{array}{r} z - (1+i) \\ \hline \big) z^2 + 2z - 2 \end{array}$$

Panel 3

Complex Numbers Graphically

$z = (x,y) = x + iy$



$\|z\| = \sqrt{x^2+y^2}$

Panel 4

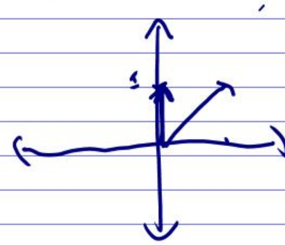
Quick Properties:

If $c \in \mathbb{R}$ and $z \in \mathbb{C}$ then $\|c \cdot z\| = |c| \, \|z\|$

If $z, w \in \mathbb{C}$ then $\|z \cdot w\| = \|z\| \cdot \|w\|$

$Re(z) \le |Re(z)| \le$

$Im(z) \le |Im(z)| \le$

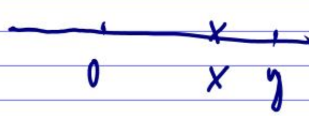


Triangle Inequality:

$\|z + w\| \le \|z\| + \|w\|$

Ex: $\|i(1+i)\| = \|i\| \, \|1+i\| = \sqrt{2}$

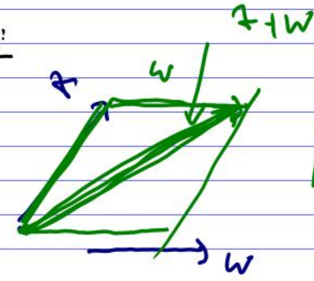
$|x+y| \le |x| + |y|$



Panel 5

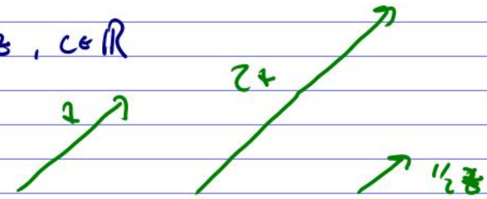
Complex Numbers Graphically:

How to add: $z+w$

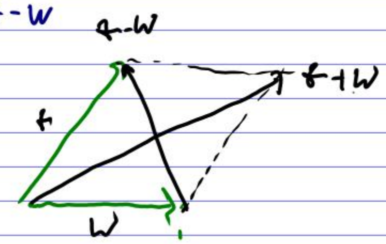


$\|z+w\| \leq \|z\| + \|w\|$

How to find $c\cdot z$, $c \in \mathbb{R}$



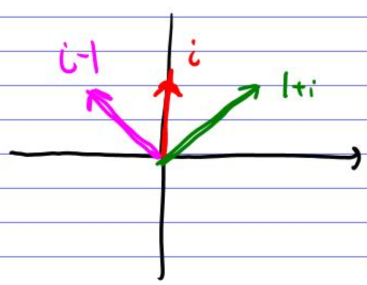
How to subtract: $z-w$



Panel 6

How to multiply: $z \cdot w$

$(1+i)\, i = i - 1$



?

Panel 7

Complex Conjugate

Def: If $z = x + iy$, then the conjugate of z is:

$$\bar{z} = x - iy$$

Ex: $\overline{3+4i} = 3-4i$

$\overline{-1-i} = -1+i$

$\overline{(1+i)(1-i)} = (1-i)(1+i)$

$\bar{2} = 2$

$Re(z) = \frac{z+\bar{z}}{2} = \frac{x+iy+x-iy}{2} = \frac{2x}{2} = x$

$Im(z) = \frac{z-\bar{z}}{2i} = \frac{x+iy-(x-iy)}{2i} = \frac{2iy}{2i} = \underline{y}$

Panel 8

Properties of Conjugates

(a) $\overline{z+w}$

(b) $\overline{z \cdot w}$

(c) $\overline{\left(\frac{z}{w}\right)}$

(d) $\overline{(\bar{z})}$

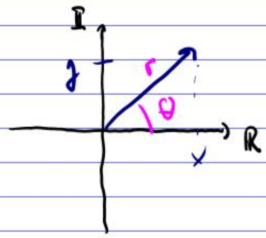
(e) $Re(z) =$

(f) $Im(z) =$

(g) $z \cdot \bar{z} = (x+iy)(x-iy) = x^2 + y^2 = \|z\|^2$

Panel 9

Still not solved: How to visualize complex multiplication



$z = x + iy$

$z = r\cos(\theta) + i\,r\sin(\theta) =$

$= r\left(\cos(\theta) + i\sin(\theta)\right)$

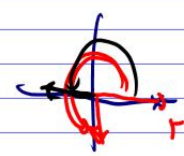
Def: Every complex number z has a length $r = \|z\|$ and angle, called $\arg(z)$

Def: $\operatorname{Arg}(z)$ is that angle of z s.t. $-\pi < \theta \leq \pi$
Principle argument.

Panel 10

Ex: $z = 5 \Rightarrow \arg(z) = 0$ or 2π ...

$\operatorname{Arg}(z) = 0$



$z = i \Rightarrow \arg(z) = \pi/2$ or $\frac{5\pi}{2}$ or ...

$\operatorname{Arg}(z) = \pi/2$

$z = -i \Rightarrow \arg(z) = \frac{3\pi}{2}$

$\operatorname{Arg}(z) = -\pi/2$

$z = -5 \Rightarrow \arg(z) = \pi$ or $-\pi$ or 3π or ...

$\operatorname{Arg}(z) = \pi$

$z = 1 - i \Rightarrow \arg(z) = \pi/4$ ...

Panel 11

Recall from Calc 2:

$\rightarrow e^x = \underline{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \quad = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

Thus: $e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots$

$= \underline{1} + ix \underbrace{- \frac{x^2}{2!}} - i\frac{x^3}{3!} + \underbrace{\frac{x^4}{4!}} + i\frac{x^5}{5!} + \dots$

$= \underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots} \; i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$

$= \cos(x) + i\sin(x)$

Panel 12

Euler's Formula: $e^{it} = \cos(t) + i\sin(t)$

Thus: Take any $z \in \mathbb{C}, z \neq 0$. Then

$\underline{\underline{z}} = x + iy = r(\cos(t) + i\sin(t)) = \underline{r\, e^{it}}$

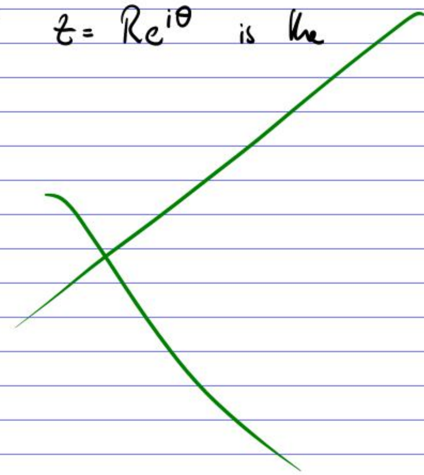
Ex $e^{i\frac{\pi}{2}} = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i$

$5e^{i\pi} = 5(\cos(\pi) + i\sin(\pi)) = \underline{\underline{-5}}$

$1 + i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

Panel 13

Geometrically, $z = Re^{i\theta}$ is the



Panel 14

How does this help visualizing multiplication?

$z = R_1 e^{it_1}$

$w = R_2 e^{it_2}$

$$z \cdot w = (R_1 e^{it_1})(R_2 e^{it_2}) = R_1 R_2 (\cos(t_1) + i\sin(t_1))(\cos(t_2) + i\sin(t_2))$$

$$= R_1 R_2 [(\cos(t_1)\cos(t_2) - \sin(t_1)\sin(t_2)) + i(\cos(t_1)\sin(t_2) + \sin(t_1)\cos(t_2))]$$

$$= R_1 R_2 (\cos(t_1+t_2) + i\sin(t_1+t_2)) = R_1 R_2 e^{i(t_1+t_2)}$$

To mult. two complex numbers: !!!

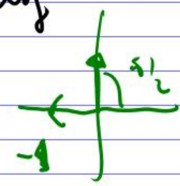
(a) mult. their length

(b) add angles!!!

Panel 15

Ex: length 1, angle $\pi/2 + \pi/2 = \pi$

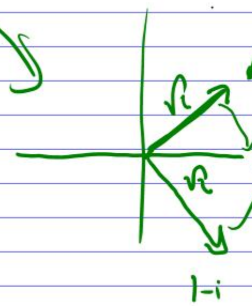
$i^2 = i \cdot i = -1$ ⇒ show graphically



Find $(1+i)(1-i)$ graphically / algebraically

Find $(1+i)^4$

Find $(\sqrt{3}+i)^7$

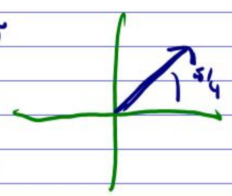


$2 \cdot e^{i0} = \underline{\underline{2}}$

$(\sqrt{3}+i)^7$

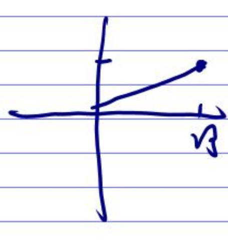
Panel 16

$(1+i)^4 = (\sqrt{2})^4 e^{i\pi}$



$= \underline{\underline{-(\sqrt{2})^4}}$

$(\sqrt{3}+i)^7 = \left(2e^{i\frac{\pi}{6}}\right)^7 = 2^7 e^{i\frac{7\pi}{6}} = 2^6 e^{i\frac{6\pi}{6} + i\frac{\pi}{6}} \cdot 2$



$\tan(\theta) = \sqrt{3}$

$\Rightarrow \theta = \pi/6$

$= 2^6 e^{i\pi} e^{i\pi/6}\, 2$

$= 2^6 \left(-2e^{i\pi/6}\right)$

$= \underline{\underline{-2^6 (\sqrt{3}+i)}}$

Panel 17

Recall: (Euler's Thm) $e^{it} = \cos(t) + i\sin(t)$

Euler's Formula: $e^{i\pi} + 1 = 0$