Complex Analysis Exam 2

*This is a take-home exam. You may use the book or your notes as you wish, but you **must** complete each problem on your own. Show all your work (and be neat). Due: last day of finals – **no** exceptions!*

1. Perform the following integrations along the indicated contours. You can use any method you like.

   a) $\int_C \frac{e^z}{z-2}\,dz$, C the unit circle $|z|=1$ b) $\int_C \frac{e^{iz}}{z^3}\,dz$, C the square with corners 1, i, -1, and –i.

   c) $\int_C \frac{\cos(2z)}{z(z-2)}\,dz$, C the circle $|z-3|=2$ d) $\int_C \frac{2z+1}{z^2(z^2+1)}\,dz$, C the circle $|z|=2$

   e) $\int_C z^4 e^{2/z}\,dz$ C the circle $|z-i|=42$

2. Find the Taylor series for each given function centered at the point $z_0=0$. Specify the radius of convergence for each series.

   a) $f(z)=z^3\cos(z^2)$ b) $f(z)=\frac{z}{3-2z}$ c) $f(z)=\ln(1+z)$

3. Find a Laurent series for the given function centered at the given point z_0 that converges in the specified domain.

   a) $f(z)=z^3e^{\frac{1}{z}}$, $z_0=0$, convergent in domain including $z=1$

   b) $f(z)=\frac{1}{3-4z+z^2}$, $z_0=0$, convergent in domain including $z=2$

4. Consider the function $f(z)=\frac{e^z}{(3-z)(z^2-1)}$ If you were to find the Laurent series centered at $z=i$ converging in the largest annulus $r<|z-i|<R$ including the point $z=2$, then what are r and R?

5. Each of the following functions has one or more isolated singularity. Identify each singularity and classify it as removable, pole, or essential. If it is a pole, find its order. Also, find the residue at each singularity.

   a) $f(z)=z^3\cos\left(\frac{1}{z}\right)$ b) $g(z)=\frac{z}{(z^2+1)(z+1)^2}$ c) $h(z)=\frac{e^z-1}{z}$

6. Use the (complex) Residue Theorem to evaluate $\int_{-\infty}^{\infty}\frac{x^2}{4+x^4}\,dx$. Make sure to justify each step. *Hint*: the answer is $\pi/2$

Extra credit: An analytic function $f(z)$ is said to have a *zero of order m* at z_0 if $f(z)=\sum_{n=m}^{\infty}a_n(z-z_0)^n$, i.e. the first non-zero coefficient in the Taylor series for *f* is a_m. Suppose $f(z)$ is analytic near z_0 with a zero of order k at z_0 Show that $\frac{f'(z)}{f(z)}$ has a pole of order 1 at z_0. *Hint:* factor what you can from *f(z)*, then work out *f'(z)/f(z)* and use a theorem on what it means to have a pole of order *m* (or 1 in our case).