Complex Practice Exam 1

This practice exam contains ***sample*** *questions. The actual exam will have fewer questions, and may contain questions not listed here.*

1. Be prepared to explain the following concepts, definitions, or theorems:
   - A complex number, polar coordinates, rectangular coordinates
   - Add, Multiply, Sub, Div, Conjugate, abs Value, graphical interpretations of these
   - Complex roots
   - Mapping properties of complex functions
   - Arg(z) and arg(z)
   - The limit of a complex function f(z) as z approaches c is L
   - Continuity of a complex function f(z) at a point z = c
   - The complex derivative of a function f(z)
   - Analytic function and Entire function
   - CR equations
   - f(z) analytic & f'(z) = 0, f(z) analytic & f-conjugate analytic, f(z) analytic and |f(z)| constant
   - Harmonic function and harmonic conjugate of a function u (incl. how to find)
   - e^z, sin(z), cos(z), log(z), and Log(z)
   - Euler's Formula, De Moivre's Formula
   - Complex parametric functions z(t), their integrals and derivatives
   - Different paths (line segments and circles)
   - Contour Integrals

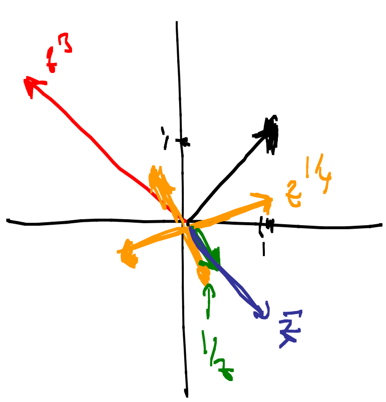
see book or notes

2. Describe the set of points z such that
   (a) $\operatorname{Re}(z) = 1$ $\Rightarrow$ $z = x+iy$, $\operatorname{Re}(z) = 1 \Leftrightarrow x = 1 \Rightarrow$ vertical line through $x=1$.

   (b) $|z-1| = 2$ $\sqrt{(x-1)^2+(y)^2} = 2 \Rightarrow (x-1)^2+y^2 = 4 \Rightarrow$ circle, center $z = 1$, radius 2

   (c) $Arg(z) = \dfrac{\pi}{4}$ $z = re^{it}$, $\operatorname{Arg}(z) = \pi/4 \Rightarrow z = re^{i\pi/4}$ is main diagonal, pos. part, like a ray

3. Let $z = 1+i$. Draw, in one coordinate system, $\bar{z}$, $\dfrac{1}{z}$, z^3, and $z^{\frac{1}{4}}$



$\bar z = 1-i$

$\frac{1}{z} = \frac{\bar z}{\|z\|} = \frac{1}{\|z\|}\cdot \bar z$

$z^3 = |\sqrt{2}e^{i\pi/4}|^3 = 2^{3/2}e^{i3\pi/4}$

$z^{1/4} = (\sqrt{2})^{1/4}\cdot(e^{i\pi/4})^{1/4}$

4. Compute/simplify the following and find real and imag parts:

a) $\left|\overline{(1+i)}(1-i)i\right|$ $= \left|(1-i)^2 i\right| = |1-i|^2 \cdot |i| = \left(\sqrt{2}\right)^2 \cdot 1 = \underline{\underline{2}}$ $\operatorname{Re}(\;) = 2$, $\operatorname{Im}(\;) = 0$

(b) $\dfrac{i(1+i)^3}{(1-i)^2}$ $= \dfrac{e^{i\frac{\pi}{2}}\left[\frac{1}{\sqrt{2}}\left(e^{i\pi/4}\right)\right]^3}{\left(\frac{1}{\sqrt{2}}e^{-i\pi/4}\right)^2} = \dfrac{2^{1/2}}{2^{3/2}} \cdot e^{i\left(\pi/2 + 3\pi/4 + 2\pi/4\right)} = \dfrac{1}{\sqrt{2}} e^{i\left(7\pi/4\right)} = \underline{\underline{1-i}}$

$\operatorname{Re}(\;) = 1$, $\operatorname{Im}(\;) = -1$

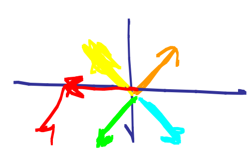
(c) $(1+i)^6$ $\left(\frac{1}{\sqrt{2}}\right)^6 \left(e^{i\pi/4}\right)^6 = 2^3 e^{i\frac{6\pi}{4}} = 2^3 e^{i3\pi/2} = -8i$ $\operatorname{Re}(\;) = 0$, $\operatorname{Im}(\;) = -8$

(d) $\dfrac{2+2i}{-\sqrt{3}+i}$ $\dfrac{-\sqrt{3}-i}{-\sqrt{3}-i} = \dfrac{-2\sqrt{3}+2-2\sqrt{3}i-2i}{3+1}$ $\Rightarrow \operatorname{Re}(\;) = \frac{1}{4}\left(2-2\sqrt{3}\right)$

$\operatorname{Im}(\;) = -\frac{1}{4}\left(2+2\sqrt{3}\right)$

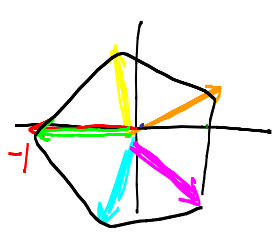
5. Find the fourth roots of -1, i.e. $\sqrt[4]{-1}$, and display them graphically.

$(-1)^{1/4} = \left(e^{i\pi}\right)^{1/4} = e^{i\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right)}$



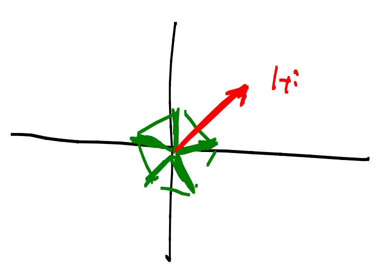
Do the same for the fifth roots of -1

$(-1)^{1/5} = \left(e^{i\pi}\right)^{1/5} = e^{i\frac{2k\pi}{5}}$



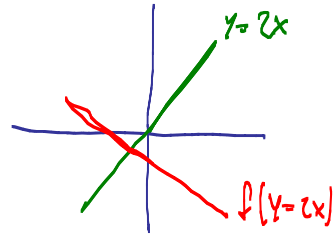
and of (1+i).

$\left[\frac{1}{\sqrt{2}} e^{i\pi/4}\right]^{1/5} = 2^{1/10} e^{i\left(\frac{\pi}{20} + \frac{2k\pi}{5}\right)}$



6. Find the image of the line $y = 2x$ under the map $f(z) = iz - 1$.

$y = 2x \Leftrightarrow r(t) = t + i2t$



$\Rightarrow f(\text{line}) = f(t + i2t) = i(t + 2ti) - 1 = it - 2t - 1 \Rightarrow \begin{array}{l} y = t \\ x = -2t - 1 = -2y - 1 \end{array} \Rightarrow y = -\frac{1}{2}(x+1)$

What is the image of the unit circle under the same map?

unit circle: $z = e^{it}$, $t \in [0, 2\pi]$ $\Rightarrow$ $f(z) = ie^{it} - 1 = e^{i(t + \pi/2)} - 1$ is another circle shifted right by 1 and starting at a $\pi/2$ angle

7. Consider the following questions, involving limits and continuity of complex functions. Remember that limits can be taken in different directions, and for complicated limits there is l'Hospital's rule.

a) If $f(z) = \dfrac{x - iy}{x + iy}$, then f is clearly undefined at $z = 0$. Can you define *f(0)* in such a way that the new function is continuous at every point in the complex plane?

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x - iy}{x + iy}$: $(x,0) \to (0,0)$: $\lim = 1$; $(0,y) \to (0,0)$: $\lim = -1$

$\Rightarrow$ limit does not exist, so there is no way to make $f(z)$ continuous at $z = 0$

b) Say $f(z) = \dfrac{z^9 + z - 2i}{z^{15} + i}$ Can you define *f(i)* in such a way that the new function is continuous at every point in the complex plane?

$\lim\limits_{z\to i} \dfrac{z^9 + z - 2i}{z^{15} + i} = \dfrac{0}{0} \overset{\text{l'Hospital}}{=} \lim\limits_{z\to i} \dfrac{9z^8 + 1}{15z^{14}} = \dfrac{10}{-15} = \underline{\underline{-\dfrac{2}{3}}}$ so: $f = \begin{cases} \dfrac{z^9 + z - 2i}{z^{15} + i}, & z \neq i \\ -2/3, & z = i \end{cases}$

is cont. at $z = i$.

c) Find $\lim\limits_{z\to 1} \dfrac{1 + z^6}{1 + z^{10}}$, $= \dfrac{2}{2} = \underline{\underline{1}}$

$\lim\limits_{z\to i} \dfrac{1 + z^6}{1 - z^{10}} = \dfrac{0}{2} = \underline{\underline{0}}$

$\lim\limits_{z\to i} \dfrac{1 + z^6}{1 + z^{10}} = \dfrac{0}{0} = \lim\limits_{z\to i} \dfrac{6z^5}{10z^9} = \dfrac{6}{10} = \underline{\underline{3/5}}$

8. Consider the following questions about analytic functions.

a) If $f(z)=\dfrac{1}{\left(z^2+1\right)^2}$ then determine where, if at all, the function is analytic.

If it is analytic, find the complex derivative of f.

f is analytic $\forall z \neq \pm i$. The derivative is

$$f'(z) = \underline{-2(z^2+1)^{-3}\cdot 2z}, \quad z \neq \pm i$$

b) If $f(z)=\overbrace{x^3-3xy^2}^{u}+i\left(\overbrace{3x^2y-y^3}^{v}\right)$ then determine where, if at all, the function is analytic. If it is analytic, find the complex derivative of f.

$u_x = 3x^2-3y^2 \qquad v_y = 3x^2-3y^2$

$u_y = -6xy \qquad v_x = 6xy$

So CR are true $\forall (x,y) \Rightarrow f'(z) = 3x^2-3y^2+6xyi = \underline{3z^2}$

9. Decide which of the following functions are analytic, and in which domain they are analytic. If a function is analytic, find its complex derivative:

(a) $f(z)=\dfrac{e^z+1}{e^z-1}$ analytic iff $e^z \neq 1 \Rightarrow \underline{z \neq 0, \pm 2\pi i, \pm 4\pi i, \pm 8\pi i, \ldots}$

$$f'(z) = \frac{e^z(e^z-1)-(e^z+1)e^z}{(e^z-1)^2} = \frac{-2e^z}{(e^z-1)^2}$$

(b) $f(z)=\underbrace{x^3}_{u}+\underbrace{3ix^2y}_{v}-3xy^2+x-iy^3+iy$

$u = x^3-3xy^2+x \qquad v = 3x^2y-y^3+y$

$u_x = 3x^2-3y^2+1 \qquad v_y = 3x^2-3y^2+1$

$u_y = -6xy \qquad v_x = 6xy$

So CR are true $\forall x,y$ $\Rightarrow$ analytic everywhere, and

$$f' = 3x^2-3y^2+1+i\,6xy = \underline{3z^2+1}$$

10. Consider the function $u(x, y) = e^x \sin(y)$. Is it harmonic ? If so, find its harmonic conjugate.

$\Rightarrow u_{xx} + u_{yy} = 0$ is harmonic

$u = e^x \sin(y)$

$u_x = e^x \sin(y)$, $u_{xx} = e^x \sin(y)$

$u_y = e^x \cos(y)$ $u_{yy} = -e^x \sin(y)$

$u_x = e^x \sin(y) = v_y$

$\Rightarrow v = -e^x \cos(y) + C(x) \Rightarrow v_x = -e^x \cos(y) + C'(x)$
$= -u_y = -e^x \cos(y)$

$\Rightarrow C'(x) = 0 \Rightarrow f = e^x \sin(y) - e^x \cos(y)\, i + C$

Do the same for (a) $u(x, y) = x^3 - 2xy + xy^3$

$u_x = 3x^2 - 2y - y^3$ $u_{xx} = 6x$

$u_y = -2x + 3xy^2$ $u_{yy} = 6xy$

not harmonic

(b) $u(x, y) = e^y \cos(x)$

$u_x = -e^y \sin(x)$, $u_{xx} = -e^y \cos(x)$

$u_y = e^y \cos(x)$, $u_{yy} = e^y \cos(x)$

0

$\Rightarrow u$ is harmonic

$u_x = -e^y \sin(x) = v_y \Rightarrow v = -e^y \sin(x) + C(x)$

$\Rightarrow v_x = -e^y \cos(x) + C'(x) \overset{?}{=} -u_y = -e^y \cos(x)$

$\Rightarrow C'(x) = 0 \Rightarrow v(x,y) = -e^y \sin(x) + C$

$f(z) = e^y \cos(x) - i e^y \sin(x) + C$

11. Please find the following numerical answers:

(a) e^{2+2i}, $= e^2 \cdot (\cos(2) + i \sin(2))$

(b) $\cos(\pi + i)$, $= \frac{1}{2}\left(e^{i(\pi+i)} + e^{-i(\pi+i)}\right) = \frac{1}{2}\left(e^{i\pi} e^{-1} + e^{-i\pi} e^{1}\right) = \frac{1}{2}\left(-e^{-1} - e^{1}\right) = -\cosh(1)$

(c) $\sin\left(i - \frac{\pi}{2}\right)$, $= \frac{1}{2i}\left(e^{i(i-\pi/2)} - e^{-i(i-\pi/2)}\right) = \frac{1}{2i}\left(e^{-1} e^{-\frac{\pi}{2}i} - e^{1} e^{\frac{\pi}{2}i}\right) = \frac{1}{2i}\left(-i e^{-1} - i e^{1}\right) =$
$= -\frac{1}{2}\left(e^{1} + e^{-1}\right) = -\cosh(1)$

(d) $\log(-2)$, $= \log(2e^{i\pi}) = \ln(2) + i(\pi + 2k\pi)$

(e) $Log(1+i)$ $= Log\left(\sqrt{2}\, e^{i\pi/4}\right) = \ln(\sqrt{2}) + i\,\pi/4 = \frac{1}{2}\ln(2) + i\,\pi/4$

14 Solve the following equations for z.

(a) $z^4+1=0$,

$z = \sqrt[4]{-1}$, see above

(b) $|e^{2z}|=3$, $e^{2x}=3 \Rightarrow z = \frac{\ln(3)}{2} + iy$, any y.

(c) $\sin(z)=3i$, $\frac{1}{2i}(e^{iz}-e^{-iz}) = 3i \Rightarrow e^{iz}-e^{-iz} = -6 \quad |\cdot e^{iz} \Rightarrow (e^{iz})^2 - 1 = -6e^{iz}$

$\Rightarrow (u)^2+6u-1=0 \Rightarrow u = -3 \pm \sqrt{10} = e^{iz} \Rightarrow \ln(-3\pm\sqrt{10}) = iz \Rightarrow z = -i\cdot\ln(-3\pm\sqrt{10})$

check with Maple: $\sin(-i\cdot\ln(-3\pm\sqrt{10}))$ check.

(d) $e^{4z}=1$, (e) $\cos(z)=i\sin(z)$

$e^{4z}=1$

$\Rightarrow z = i\left(\frac{\pi}{2}+\frac{2k\pi}{4}\right)$

$\frac{1}{2}(e^{iz}+e^{-iz}) = i\frac{1}{2i}(e^{iz}-e^{-iz}) \Leftrightarrow \cancel{e^{iz}}+e^{-iz} = \cancel{e^{iz}}-e^{-iz}$

$\Rightarrow 0 = e^{-iz} \Rightarrow$ no solution

15 Use the definition of derivative to show that the functions $f(z)=\mathrm{Re}(z)$ is nowhere differentiable.

$f(z) = \mathrm{Re}(z) = x \quad \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{(x,y)\to(x_0,y_0)} \frac{x-x_0}{x-x_0+i(y-y_0)}$

~~$x=x_0$:~~ lim is zero

$y=y_0$: lim is 1

$\Rightarrow$ limit does not exist!

Use the CR equations to show that the function $f(z)=\bar{z}$ is nowhere differentiable.

$f(z) = \bar{z} = x - iy \quad u_x = 1, \; v_y = -1 \Rightarrow$ no match so

f is nowhere diff'ble

$(uv)_x = u_x v + u v_x \Rightarrow (uv)_{xx} = u_{xx}v + u_x v_x + u_x v_x + u v_{xx}$

because u, v are harmonic

Show that if v is the harmonic conjugate of u, then the product u v is harmonic.

Know: u harmonic $\Rightarrow u_{xx}+u_{yy}=0$

v harmonic $\Rightarrow v_{xx}+v_{yy}=0$

$u_x = v_y$ and $u_y = -v_x$

also $(uv)_{yy} = u_{yy}v + 2u_y v_y + u v_{yy}$

thus $(uv)_{xx}+(uv)_{yy} = u_{xx}v + 2u_x v_x + u v_{xx} + u_{yy}v + 2u_y v_y - u v_{yy} = 0$

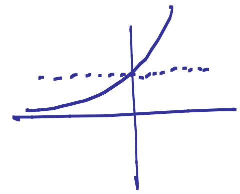
$u_{xx}+u_{yy}=0$ 0 0

CR equations

$\Rightarrow$ so harmonic!

16 Show that $\left|e^{z}\right| \leq 1$ if $\operatorname{Re}(z) \leq 0$

$|e^z| = |e^x e^{iy}| = e^x < 1$ if $\operatorname{Re}(z) = x < 0$



17 State De Moivre's formula. Then use it to prove the trig identity $\sin(2x) = 2\sin(x)\cos(x)$

$(e^{it})^2 = (e^{2it})$ or $(\cos t + i\sin t)^2 = \cos 2t + i\sin(2t)$

$\Rightarrow \cos^2(t) - \sin^2(t) + 2i\cos(t)\sin(t) = \cos(2t) + i\sin(2t)$

$\Rightarrow \cos^2(t) - \sin^2(t) = \cos(2t)$ and $2\cos(t)\sin(t) = \sin(2t)$

18 Show that the function e^{iz} is periodic with period 2π

$e^{i(z+2\pi)} = e^{iz}e^{2\pi i} = e^{iz}\cdot 1 = e^{iz}$

19 Show that the function *sin(z)* is unbounded

$\lim_{t\to\infty} \sin(it) = \lim_{t\to\infty} \left|\frac{1}{2i}\left(e^{i(it)} - e^{-i(it)}\right)\right| = \lim_{t\to\infty} \frac{1}{2}\left|e^{-t} - e^{t}\right| = \infty$

20 Show that the function $f(z) = z\bar{z} + z + \bar{z} + 2x$ cannot be an analytic function.

$f(z) = z\bar{z} + z + \bar{z} + 2x = x^2 + y^2 + 2x + 2x = x^2 + y^2 + 4x$

$\Rightarrow u_x = 2x + 4 \quad , v_y = 0 \quad \Rightarrow x = -2$

$u_y = 2y \quad -v_x = 0 \quad \Rightarrow y = 0$

CR equations are true only at a point, not in a full nbhd.

$\Rightarrow$ not analytic

21 Prove that $\sin^2(z)+\cos^2(z)=1$ (Hint: take the derivative of $f(z)=\sin^2(z)+\cos^2(z)$)

$$\frac{d}{dz}\left(\sin^2(z)+\cos^2(z)\right)=2\sin(z)\cos(z)+2\cos(z)(-\sin(z))=0$$

$$\Rightarrow \sin^2(z)+\cos^2(z)=\text{constant.} \quad =1 \quad (\text{if } z=0)$$

22 Prove the following theorem: If *f(z)* is an analytic function with values that are always imaginary, then the function must be constant.

use CR equations

23 Prove the following theorem: if is a harmonic function in an open set U (i.e. h is twice continuously differentiable and $\frac{\partial^2}{\partial x^2}h+\frac{\partial^2}{\partial y^2}h=0$ in the open set U), then the complex function $f(z)=\frac{\partial}{\partial x}h(x,y)-i\frac{\partial}{\partial y}h(x,y)$ is an analytic function in U.

$f=h_x-ih_y \Rightarrow u=h_x$ and $v=-h_y$

check CR: $u_x=h_{xx}$ $\quad v_y=-h_{yy}$ and $h_{xx}+h_{yy}=0$ (h harmonic)

$\Rightarrow u_x=v_y$

$u_y=h_{xy}$ $\quad -v_x=h_{yx}$ and $h_{xy}=h_{yx}$ (h 2x cont. diff.)

$\Rightarrow u_y=-v_x$ $\quad \Rightarrow$ CR check so f is analytic.

24 Find complex parametric functions representing the following paths:
(a) a straight line from –i to i,

$$z(t)=-i+t(2i),\quad t\in[0,1]$$

(b) the right half of a circle from –i to i,

$$z(t)=e^{it},\quad t\in[-\pi/2,\pi/2]$$

(c) a straight line from -1 – 2i to 3 + 2i

$z(t) = (-1-2i) + t(3+2i-(-1-2i)) = (-1-2i) + t(4+4i)$, $t \in [0,1]$

(d) a circle centered at 1+i of radius 2 ~~ccw~~

$z(t) = 2e^{it} + 1+i$, $t \in [0, 2\pi]$

25 Evaluate

a. z'(t) for $z(t) = \cos(2t) + i\sin(2t)$

$z'(t) = -2\sin(2t) + 2i\cos(2t)$

b. $\int_0^{\pi} z(t)dt$ for $z(t) = (5+4i)e^{3it}$

$\int_0^{\pi} (5+4i)e^{3it}dt = (5+4i)\int_0^{\pi} e^{3it}dt = (5+4i)\frac{1}{3i}e^{3it}\Big|_0^{\pi} = \frac{1}{3i}(5+4i)(e^{3\pi i}-1) = -\frac{2}{3i}(5+4i) = \frac{2}{3}i(5+4i)$

26 Evaluate

a. $\int_{\gamma} iz^2 + 3dz$ where γ is a line segment from -1-i to 1+i

$z(t) = -1-i + t(1+i-(-1-i)) = -1-i+2(1+i)t$

$\Rightarrow \int_{\gamma} iz^2 + 3dz = \int_0^1 \left[i(-1-i+2(1+i)t)+3\right](2(1+i))dt = 6+6i$ ← Maple

b. $\int_{\gamma} \frac{1}{\bar{z}}dz$ where γ is a circle radius 2 centered at the origin

$z(t) = 2e^{it}$, $t \in [0, 2\pi]$

$\int_0^{2\pi} \frac{1}{2e^{-it}} \cdot 2ie^{it}dt = i\int_0^{2\pi} e^{2it}dt = i\frac{1}{2i}e^{2it}\Big|_0^{2\pi} = 0$