Panel 1

$\int_C f(z)\,dz$, C closed.

Cauchy-Goursat: f analytic inside C

Cauchy-Int. $f(z) = \frac{g(z)}{z-z_0} = 2\pi i\, g(z_0)$

Gen. Cauchy-Int. $f = \frac{g}{(z-z_0)^n} = 2\pi i\, \frac{g^{(n-1)}(z_0)}{(n-1)!}$

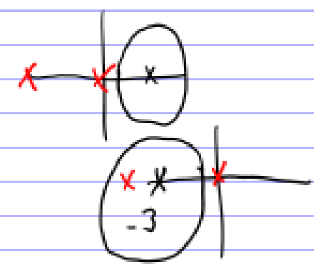
Residue Thm

$$\int_C \frac{1}{z^3(z+4)}\,dz$$

a) $C: |z-2|=1$: 0

b) $C: |z+3|=2$: $2\pi i\, \frac{1}{(-4)^3}$

c) $C: |z-2|=3$



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Panel 2

$$\int_{|z-2|=3} \frac{1}{z^3(z+4)}\,dz = 2\pi i\, \mathrm{Res}(f,0)$$

Pole, order 3. $\lim_{z\to 0} \frac{d^2}{dz^2} \cancel{z^3} \frac{1}{\cancel{z^3}(z+4)} \frac{1}{2!} = 2(z+4)^{-3}\Big|_{z=0} \frac{1}{2!} = \frac{2}{4^3\, 2}$

↑

if $z^3 \cdot f(z) \neq 0$ at $z=0$

$$\int_{|z|=10} \frac{1}{z^3(z+4)}\,dz = 2\pi i\left(\mathrm{Res}(f,0) + \mathrm{Res}(f,-4)\right)$$

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Panel 3

$$\int_{|z|=3} \frac{z^3 e^{1/z}}{1+z^3}\,dz = 2\pi i\left[\mathrm{Res}(f,0) + \underline{\mathrm{Res}(f,-1)} + \mathrm{Res}(f, e^{i\pi/3}) + \mathrm{Res}(f, e^{i5\pi/3})\right]$$

$$\mathrm{Res}(f,-1) = \left.\frac{z^3 e^{1/z}}{\cancel{(z+1)}(z-e^{i\pi/3})(z-e^{i5\pi/3})}\right|_{z=-1} \qquad \frac{(-1)^3 e^{-1}}{(-1-e^{i\pi/3})(-1-e^{i5\pi/3})}$$

pole, order 1

$$\lim_{z\to -1}(1+z)\frac{z^3 e^{1/z}}{1+z^3} = \lim_{z\to -1}\frac{(z^3+z^4)e^{1/z}}{1+z^3} = \left.\frac{(3z^2+4z^3)e^{1/z} + (z^3+z^4)e^{1/z}\left(-\frac{1}{z^2}\right)}{3z^2}\right|_{z=-1}$$

$$= \frac{(3-4)e^{-1} + \cancel{(-1+1)}}{3\cdot 1} = -\frac{e^{-1}}{3}$$

Panel 4

$$\frac{z^3 e^{1/z}}{1+z^3}, \quad \mathrm{Res}(f,0), \quad 0 \text{ is essential!} \qquad \frac{1}{1-(-z^3)}$$

$$z^3\left(1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2\frac{1}{2!} + \left(\frac{1}{z}\right)^3\frac{1}{3!} + \left(\frac{1}{z}\right)^4\frac{1}{4!} + \dots\right)\left(\frac{1}{1+z^3}\right)$$

$$\left(\cancel{z^3} + \cancel{z^2} + \cancel{z\frac{1}{2!}} + \cancel{\frac{1}{3!}} + \frac{1}{z}\frac{1}{4!} + \cancel{\frac{1}{z^2}}\frac{1}{5!} + \cancel{\frac{1}{z^3}}\frac{1}{6!} + \dots\right)\left(1 - z^3 + z^6 - z^9 + z^{12} - \dots\right)$$

$$\frac{1}{z}\left(\frac{1}{4!} - \frac{1}{7!} + \frac{1}{10!} - \frac{1}{13!} + \dots\right)$$

Panel 5

$$\cancel{\int_0^{2\pi}\frac{4}{2+3\sin(t)}\,dt}$$ improper because $2+3\sin(t)=0$ for some t

$$\int_0^{2\pi}\frac{4}{3+2\sin(t)}\,dt = \int_{|z|=1}\frac{4}{3+\frac{1}{i}\left(z-\frac{1}{z}\right)}\cdot\frac{1}{iz}\,dz = \frac{1}{i}\int\underline{\frac{4}{3z-iz^2+i}}\,dz$$

$z=e^{it}$, $\sin(t)=\frac{1}{2i}\left(e^{it}-e^{-it}\right)=\frac{1}{2i}\left(z-\frac{1}{z}\right)$

$dz = ie^{it}dt$

$= i\,z\,dt$

$\Rightarrow dt = \frac{dz}{iz}$

$-iz^2+i+3z=0 \quad |\cdot i$

$z^2+3iz-1=0$

$$z_{1/2}=\frac{-3i\pm\sqrt{-9+4}}{2}=\frac{-3i\pm\sqrt{-5}}{2}=\frac{-3i\pm i\sqrt{5}}{2}$$

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Panel 6

Sing. $\cancel{\frac{-3i-i\sqrt{5}}{2}}=z_1$, $\frac{-3i+i\sqrt{5}}{2}=z_2$

outside $|z|=1$

$$\mathrm{Res}(f,z_2)=\lim_{z\to z_2}(z-z_2)\frac{4}{3z-iz^2+i}=\frac{4}{3-2iz}\Big|_{z_2}=\frac{4}{3-\cancel{2}i\frac{(-3i+i\sqrt{5})}{\cancel{2}}}$$

$$=\frac{4}{3-3+\sqrt{5}}=\underline{\underline{\frac{4}{\sqrt{5}}}}$$

$$\int \_\_\_\, dt = 2\pi i\cdot\frac{1}{i}\cdot\frac{4}{\sqrt{5}}=\underline{\underline{\frac{8\pi}{\sqrt{5}}}}$$

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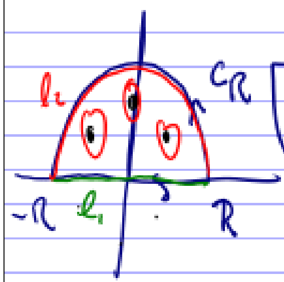
Panel 7

$$\int_{-\infty}^{\infty}\frac{1}{1+x^6}\,dx \overset{?}{=} \lim_{R\to\infty}\int_{C_R}\frac{1}{1+z^6}\,dz = \lim_{R\to\infty}\left[\int_{\ell_1}\frac{1}{1+z^6}\,dz + \cancelto{0}{\int_{\ell_2}\frac{1}{1+z^6}\,dz}\right]$$



$$\boxed{2\pi i \sum \mathrm{Res}(f, z_i) = \int_{-\infty}^{\infty}\frac{1}{1+x^6}\,dx}$$

$$\left|\int_{\ell_2}\frac{1}{1+z^6}\,dz\right| = \frac{\pi R}{1-R^6} \xrightarrow[R\to\infty]{} 0$$

$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx$$

$$\left|\int \frac{z^3}{1+z^4}\,dz\right| \le \frac{\pi R\cdot R^3}{1-R^4} = \frac{\pi R^4}{1-R^4}$$

$$\cancel{\int_{-\infty}^{\infty}\frac{x^3}{1+x^4}\,dx}$$

sing. on $\mathbb{R}$

$$\cancel{\int_{-\infty}^{\infty}\frac{1}{1+x^3}\,dx}$$

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Panel 8

Thm: $$\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\,dx = 2\pi i \sum \mathrm{Res}(f, z_i), \quad f(z) = \frac{p(z)}{q(z)}$$

if p, q are polynomials, $q(x)\neq 0$ for $x\in\mathbb{R}$, $\deg(p) < \deg(q) - 1$

HW: proof

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