Panel 1

Residue Thm: f analytic inside closed curve C except at $z_1, z_2, \dots, z_n$. Then

$$\oint_C f(z)\,dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f, z_i)$$

Singularities: $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-3}}{(z-z_0)^3} + \dots$

removable: $a_{-n} \equiv 0 \quad \forall n$

pole order n: $a_{-n} \neq 0$, $a_{-j} = 0$, $j = 0, 1, \dots, n-1$

essential: inf many $a_n = 0$

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Panel 2

Finding Residues Thm: f has pole of order m at z_0 iff:

$$f(z) = \frac{g(z)}{(z-z_0)^m} \quad \text{for some } g \text{ analytic at } z_0,\ g(z_0) \neq 0$$

Moreover: $m = 1$: $\mathrm{Res}(f, z_0) = g(z_0)$

$m > 1$: $\mathrm{Res}(f, z_0) = \dfrac{\frac{d^{m-1}}{dz^{m-1}} g(z_0)}{(m-1)!}$

f has pole, order $m \Rightarrow$ $f(z) = \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0)^2 + \dots$

$$= \frac{1}{(z-z_0)^n} \underbrace{\left(a_{-n} + a_{-(n-1)}(z-z_0) + \dots + a_{-1}(z-z_0)^{n-1} + \dots \right)}_{g(z),\ g(z_0) = a_{-n}}$$

$a_{-1} = b_{n-1} = \dfrac{g^{(n-1)}(z_0)}{(n-1)!}$

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Panel 3

Example: Find all residues of

a) $f(z) = \dfrac{z^2+3z-4}{(z-1)^4(z+2)}$ b) $g(z) = \dfrac{z^4 \sin(1/z^2)}{(2z-1)}$

Res$(f,-2)$ in pole, order 1: $f(z) = \dfrac{g(z)}{(z+2)} =$

$g(z) = \dfrac{z^2+3z-4}{(z-1)^4}$, $g(-2) = \dfrac{-6}{3^4}$

Res$(f,1)$ in pole order 4: $f(z) = \dfrac{g(z)}{(z-1)^4}$, $g(z) = \dfrac{z^2+3z-4}{z+2}$, $g(1) = 0$ NO

order 3: $f(z) = \dfrac{\cancel{(z-1)}(z+4)}{(z-1)^4(z+2)} = \dfrac{z+4}{(z-1)^3(z+2)} = g(z)/(z-1)^3$, $g(z) = \dfrac{z+4}{z+2}$, $g(1) = \dfrac{5}{3}$

$g''(1)/2 = \text{Res}(f,1)$

Panel 4

$f(z) = \dfrac{z^4 \sin(1/z^2)}{(2z-1)}$

Res$(f, z=1/2)$ in pole order 1: $f(z) = \dfrac{g(z)}{(z-1/2)}$, $g(z) = z^4 \sin(1/z^2) \cdot 1/2$

$g(1) = \sin(4)/2\cdot 2^4 = \dfrac{\sin(4)}{32}$

Res$(f, z=0)$: essential sing. $\dfrac{\frac{1}{2}\, z^4\left(\frac{1}{z^2} - \frac{1}{3!z^6} + \frac{1}{5!z^{10}} - \dots\right)}{z - 1/2} =$

$-\left(z^2 - \frac{1}{3!z^2} + \frac{1}{5!z^6} - \dots\right)\frac{1}{1-z/2} = +\left(\cancel{z^2} - \cancel{\frac{1}{3!z^2}} + \frac{1}{5!z^6} - \dots\right)\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \dots\right)$

$+\left(-\frac{1}{3!2} + \frac{1}{5!2^5} - \frac{1}{7!2^9} + \dots\right)$ $\pm 1/2$

Panel 5

How to find Residues (alt. method)

f has pole, order 1 $\Rightarrow$ $f(z) = \dfrac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$

$$\Rightarrow \quad (z-z_0)\, f(z) = a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \dots$$

$$\Rightarrow \lim_{z \to z_0} (z-z_0)\, f(z) = a_{-1}$$

f has pole, order 2 $\Rightarrow$ $f(z) = \dfrac{a_{-2}}{(z-z_0)^2} + \dfrac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$

$$(z-z_0)^2 f(z) = \cancel{a_{-2}} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + \dots$$

$$\frac{d}{dz}(z-z_0)^2 f(z) = a_{-1} + a_0\, 2(z-z_0) + \dots$$

$$\lim_{z \to z_0} \tfrac{d}{dz}(z-z_0)^2 f(z) = a_{-1}$$

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Panel 6

How to find Residues

f has a pole of order k at z_0. Then

order 1: $\mathrm{Res}(f, z_0) = \lim\limits_{z \to z_0} (z-z_0)\, f(z)$

order 2: $\mathrm{Res}(f, z_0) = \lim\limits_{z \to z_0} \dfrac{d}{dz}\left[(z-z_0)^2 f(z)\right]$

order k: $\mathrm{Res}(f, z_0) = \lim\limits_{z \to z_0} \dfrac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right] \dfrac{1}{(k-1)!}$

HW

?

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Panel 7

Ex: $f(z) = \frac{1}{z(1-z)^2}$ and $\int_{C_3(0)} \frac{1}{z(1-z)^2}\,dz = 2\pi i\,(1-1) = 0$

$\text{Res}(f,0)$: $\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{1}{(1-z)^2} = 1$

$\text{Res}(f,1)$: $\lim_{z\to 1} \frac{d}{dz}\left(\cancel{(z-1)^2} \cdot \frac{1}{z\cancel{(1-z)^2}}\right) = -\frac{1}{z^2}\Big|_{z=1} = -1$

Ex: $f(z) = \cos\left(\frac{1}{z}\right)$ and $\int_{C_1} \cos\left(\frac{1}{z}\right)dz$

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Panel 8

Ex: Consider $f(z) = \frac{z^3+2z}{(z-i)^3}$. Pole? Residue?

8

Panel 9

Ex: $f(z) = \frac{1}{z(e^z - 1)}$ Pole and Residue at $z_0 = 0$

$\text{Res}(f,0) = \lim_{z\to 0} z f(z) = \lim_{z \to 0} \frac{1}{e^z - 1}$ d.n.e

Pole order 1 Not

Pole, order 2 ? : $\lim_{z\to 0} \frac{d}{dz} z^2 \frac{1}{z(e^z-1)} = \lim_{z\to 0} \frac{d}{dz} \frac{z}{e^z - 1} =$ l'Hospt

$= \lim_{z\to 0} \frac{e^z - 1 - z e^z}{(e^z-1)^2} = \lim \frac{e^z - [e^z + z e^z]}{2(e^z-1)e^z}$

$= \lim_{z\to 0} \frac{-z}{2(e^z-1)} = \lim \frac{1}{2e^z} = -\frac{1}{2}$

Panel 10

$f(z) = \frac{1}{z(e^z-1)}$

$z(e^z - 1) = z\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right) = z^2\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots\right)$

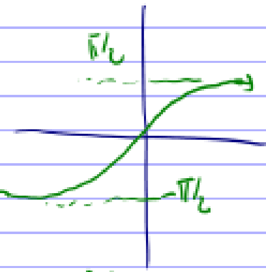
$f(z) = \frac{1}{z^2} \frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots} = \frac{1}{z^2} g(z)$, $g(0) \neq 0$ pole, order 2

$g'(z) = \frac{d}{dz}\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots\right)^{-1} = -\left(1 + \cancel{z}\dots\right)^{-2} \cdot \left(\frac{1}{2!} + \cancel{z}\dots\right)\Big|_{z=0} = -\frac{1}{2}$

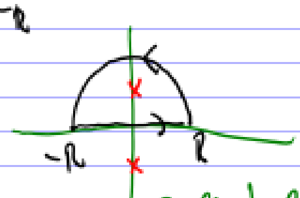
Panel 11

<u>Applications:</u> Oddly, residue thm helps to evaluate tricky real variables ints

<u>Ex:</u> $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \arctan(x)\Big|_{-R\to\infty}^{R\to\infty} = \frac{\pi}{2}+\frac{\pi}{2} = \pi$



$\int_{C_R} \frac{1}{1+z^2}\,dz = 2\pi i\,\frac{1}{2i} = \pi = \int_{-R}^{R}\frac{1}{1+x^2}\,dx + \int \frac{1}{1+z^2}\,dz \to \pi$



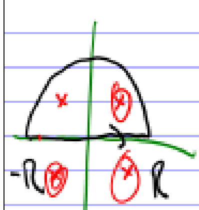
$\left|\int \frac{1}{1+z^2}\,dz\right| \le \frac{\pi R}{1-|z|^2} = \frac{\pi R}{1-R^2} \xrightarrow[R\to\infty]{} 0$

$\mathrm{Res}(f,i) = \lim_{z\to i}(z-i)\frac{1}{1+z^2} = \frac{1}{2i}$

Panel 12

<u>Ex:</u> $\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}}$

$\lim_{R\to\infty} \int_{C_R} \frac{1}{1+z^4}\,dz = 2\pi i\left(-\frac{2i}{4\sqrt{2}}\right) = \int_{-\infty}^{\infty}\frac{1}{1+x^4}\,dx$



$z_1 = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$

$z_2 = e^{i\frac{3\pi}{4}} = \frac{1}{\sqrt{2}}(-1+i)$

$z_3 = e^{i\frac{5\pi}{4}}$

$z_4 = e^{i\frac{7\pi}{4}}$

$\mathrm{Res}(f, e^{i\pi/4}) = \lim_{z\to z_1}(z-z_1)\frac{1}{1+z^4} = \frac{1}{4z_1^3}$

$\mathrm{Res}(f, e^{i3\pi/4}) = \lim_{z\to z_2}(z-z_2)\frac{1}{1+z^4} = \frac{1}{4z_2^3}$

$\frac{1}{4z_1^3} + \frac{1}{4z_2^3} = \frac{1}{4}\left(e^{-\frac{3\pi}{4}i} + e^{-\frac{9\pi}{4}i}\right) = \frac{1}{4}\left(\frac{1}{\sqrt{2}}(-1-i) + \frac{1}{\sqrt{2}}(1-i)\right) = \frac{1}{4\sqrt{2}}(-2i)$