Panel 1

Residue Thm: f analytic inside closed curve C except at $z_1, z_2, \dots, z_n$. Then

$$\int_C f(z)\,dz = 2\pi i \sum_1^n \mathrm{Res}(f, z_i)$$

Singularities:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-3}}{(z-z_0)^3} + \dots}_{\text{principle part.}}$$

removable: $a_n = 0 \quad \forall n = -1, -2, -3, \dots$

pole order n: $a_j = 0 \quad \forall j = -(n+1), -(n+2), \dots,\ a_{-n} \neq 0$

essential: inf. many $a_j \neq 0$

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Panel 2

Thm: z_0 is pole of order $m \iff f(z) = \dfrac{g(z)}{(z-z_0)^m},\ g(z_0) \neq 0$

$$\Rightarrow f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \dots \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$$

$$= \frac{1}{(z-z_0)^m} \underbrace{\left[a_{-m} + a_{-m+1}(z-z_0) + \dots a_{-1}(z-z_0)^{m-1} + a_0 (z-z_0)^m + \dots \right]}$$

$g(z)$ is analytic, $g(z_0) = a_{-m} \neq 0$

Pole order 1: $f(z) = \dfrac{1}{(z-z_0)} g(z)$, $g(z_0) \neq 0$. Moreover: $g(z_0) = a_{-1}$

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Panel 3

$$f(z) = z^2 e^{1/z} \qquad res(f, 0)$$

$$z^2 e^{1/z} = z^2 \sum_0^\infty \left(\frac{(\frac{1}{z})^n}{n!} \right) =$$

$$z^2 \sum_0^\infty \frac{1}{z^n \cdot n!} =$$

$$z^2 \left(1 + \frac{1}{z} + \frac{1}{z^2 \cdot 2!} + \frac{1}{z^3 \cdot 3!} + \cdots \right)$$

$$a_{-1} = \frac{1}{3!}$$

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Panel 4

$$f(z) = \frac{3}{z(z+2)} \qquad Res(f, 0)$$

$$\frac{3}{z} \cdot \frac{1}{z+2} \qquad \frac{1}{z+2} = \frac{1}{2} - \frac{z}{4} + \frac{z^2}{8} - \frac{z^3}{16} + \cdots$$

$$\frac{3}{z} \cdot \frac{1}{z+2} = \frac{3}{2z} - \frac{3}{4} + \frac{3z}{8} - \frac{3z^2}{16} + \cdots$$

$$\frac{1}{z} = \frac{3}{2} \qquad \boxed{Res(f, 0) = 3/2}$$

$$\frac{1}{z+2} = \frac{1}{2-(-z)} = \frac{1}{2} \frac{1}{1-\left(-\frac{z}{2}\right)}$$

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Panel 5

$g \quad \dfrac{3}{z(z+2)} \qquad Res(f,-2)$

$$\frac{-3}{2(z+2)} - \frac{3}{4} - \frac{3(z+2)}{8} - \frac{3(z+2)^2}{16}$$

$$\frac{1}{z+2} \qquad \boxed{Res(f,-2) = -3/2}$$

$$\frac{3}{z}\,\frac{1}{z+2} = \frac{1}{z+2}\,\frac{1}{(z+2)-2} = \frac{1}{z+2}\left(-\frac{1}{2}\right)\frac{1}{1-\frac{z+2}{2}} = \frac{1}{z+2}\left(-\frac{1}{2}\right)\sum\left(\frac{z+2}{2}\right)^n$$

Panel 6

Complex HW:

① Use the Residue thm to evaluate:

a) $\int_C z^6 \sin(1/z)\,dz$, C circle, center 0, radius 5

b) $\int_C \frac{3z+2}{z(z-1)}\,dz$, C circle, center 1, radius 3

c) $\int_C \frac{1}{z^4(z-3)}\,dz$, C circle, center −1, radius 2

$\Rightarrow$

Panel 7

② Identify and classify all singularities for:

a) e^z / z^4 b) $z^4 (e^{1/z} - 1)$

c) $\frac{\sin(z^2)}{z^4}$ d) $\frac{z}{(z+1)(z-3)^2}$

e) $\frac{z^2 + iz + 2}{z^2 + 1}$

Hint for d, e: f has a pole of order m at z_0 iff

$f(z) = \frac{g(z)}{(z-z_0)^m}$ where g is analytic near z_0 and $g(z_0) \neq 0$

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Panel 8

Finding Residues Thm: f has pole of order m at z_0 iff:

$f(z) = \frac{g(z)}{(z-z_0)^m}$, $g(z_0) \neq 0$. Then

$a_{-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!}$ (follows from panel #2)

Ex: $f(z) = \frac{1}{z^3(z-2)}$

$z=0$ pole order 3 : $f(z) = \frac{1}{z^3} \cdot \frac{1}{z-2}$ $g(z)$, $g(0) \neq 0$. => order 3

$z=2$ pole order $\frac{1}{z-2} \cdot \frac{1}{z^3}$ => Res(f, 2) = $\frac{1}{8}$ $a_{-1} = \frac{\frac{d^2}{dz^2}\left(\frac{1}{z-2}\right)\Big|_{z=0}}{2!} = \frac{+2(z-2)^{-3}\big|_{z=0}}{2!} = -\frac{1}{8}$ = Res

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Panel 9

$$\frac{z^2+iz+2}{z^2+1} = \frac{\overbrace{z^2+iz+2}^{g(z)}}{(z+i)(z-i)} \qquad \frac{z^2+iz+2}{(z+i)(z-i)} \; g(z)$$

$g(-i) \neq 0$

$= \frac{-2i+2}{-2i}$

$z=i$; Removable $\Rightarrow \text{Res}(f,i) = 0$

$z=-i$: Pole order 1 $\Rightarrow \text{Res}(f,-i) = g(-i) = \frac{-2i+2}{-2i}$

Panel 10

$$\frac{z}{(z^2-4)(z-4)^2}$$

$(z+2)(z-2)$

$z=2$: Pole, order 1 $\quad \text{Res}(f,2) = \left.\frac{z}{(z+2)(z-4)^2}\right|_{z=2}$

$z=-2$; Pole, order 1 $\quad \text{Res} = \left.\frac{z}{(z-2)(z-4)^2}\right|_{z=-2}$

$z=4$: Pole, order 2 $\quad \left.\left(\frac{z}{z^2-4}\right)'\right|_{z=4} = \text{Res}$

$$\Rightarrow \frac{z+2}{(z^2-4)(z-4)^2}$$

$z=2$: Pole, order 1

$z=-2$: Removable

$z=4$: Pole, order 2

$$\frac{z^2-2z-8}{(z^2-4)(z-4)^2}$$

$z=2$: Pole, order 1

$z=-2$: Pole order 1

$z=4$: Pole, order 1

Panel 11

Find $\int_0^{2\pi} \frac{1}{4+3\cos(t)}\,dt = \int_{|z|=1} \frac{1}{4+3\cdot\frac{1}{2}(z+\frac{1}{z})}\cdot\frac{1}{iz}\,dz =$

$\underline{\text{Trick}}$ $z=e^{it}$, $dz = ie^{it}dt$, $dz = iz\,dt$, $\frac{dz}{iz}=dt$

$\cos(t) = \frac{1}{2}(e^{it}+e^{-it}) = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$-i2\int_{|z|=1} \frac{1}{8z+3z^2+3}\,dz = -2i\cdot 2\pi i \sum_i \operatorname{Res}(f,z_i)$

$3z^2+8z+3=0 \iff z_{1/2} = \frac{-8\pm\sqrt{64-36}}{6} = \frac{-8\pm 2\sqrt{7}}{6} = \frac{-4\pm\sqrt{7}}{6}$

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Panel 12

$-i2\int_{|z|=1} \frac{1}{8z+3z^2+3}\,dz = -2i\cdot 2\pi i \sum_i \operatorname{Res}(f,z_i) = 4\pi \operatorname{Res}(f,z_1) = \underline{\frac{2\pi}{\sqrt{7}}}$

$3z^2+8z+3=0 \iff z_{1/2} = \frac{-8\pm\sqrt{64-36}}{6} = \frac{-8\pm 2\sqrt{7}}{6} = \frac{-4\pm\sqrt{7}}{3}$

$\underline{3z^2+8z+3 = 3(z-z_1)(z-z_2)}$, $z_1 = \frac{-4+\sqrt{7}}{3}$, $z_2 = \frac{-4-\sqrt{7}}{3}$

$\Rightarrow \frac{1}{3z^2+8z+3}$; $z=z_1$: Pole order 1, Residue $\frac{1}{3(z_1-z_2)} = \frac{1}{2\sqrt{7}}$

$z=z_2$, pole order 1 outside circle

$\frac{1}{3z^2+8z+3} = \frac{1}{3(z-z_1)(z-z_2)}$

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Panel 13

This trick applies to ~~any~~ most

$$\int_0^{2\pi} R(\sin(t), \cos(t))\,dt \quad , R \text{ is some rational function}$$

$$= \int_{|z|=1} \tilde{R}(z)\,dz = 2\pi i \sum_1^n \operatorname{Res}(\;)$$

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Panel 14

$$a^2 z^2 + bz + c = a(z-z_1)(z-z_2) \quad , z_{1/2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

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