Panel 1

Thm: If $\sum_{n=0}^{\infty} a_n (z-z_0)$ converges for $|z-z_0|<R$

then the limit function is analytic inside the circle of convergence and:

$$\frac{d}{dz} \sum_{n=0}^{\infty} a_n (z-z_0)^n = \sum \frac{d}{dz} a_n (z-z_0)^n = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

$$\int \sum_0^{\infty} a_n (z-z_0)^n = \sum \int a_n (z-z_0)^n dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-z_0)^{n+1} + C$$

1

Panel 2

Find the following Taylor series centered at $z_0=0$:

a) $f(z) = \frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)^2 = (1+z+z^2+\ldots)\cdot(1+z+z^2+\ldots) =$

$1+2z+3z^2$

$$= \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz} \sum_0^{\infty} z^n = \sum_1^{\infty} n z^{n-1} = 1+2z+3z^2+\ldots$$

b) $g(z) = \frac{2z}{(1+z^2)^2} = \frac{d}{dz} \frac{-1}{1+z^2} = \frac{d}{dz} - \sum_{n=0}^{\infty} (-1)^n z^{2n} = \underline{\underline{-\sum_{n=1}^{\infty} (-1)^n 2n z^{2n-1}}}$

c) $h(z) = \ln(1+z)$

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Panel 3

c) $\ln(1+z) = \int \frac{1}{1+z}\,dz = \int \sum_0^\infty (-1)^n z^n = \sum_{n=0}^\infty (-1)^n \frac{z^{n+1}}{n+1} =$

$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots \qquad \forall\ |z|<1$

$\Rightarrow (z=1)$: $\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ (alt. harm. series)

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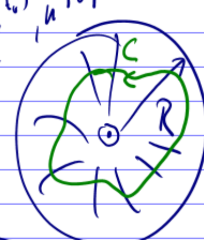
Panel 4

f analytic for $0 < |z-z_0| < R$. Then

$f(z) = \ldots + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$

$= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad$ for $0 < |z-z_0| < R$

Moreover: $\quad a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \quad \left(= \frac{f^{(n)}(z_0)}{n!},\ n>0\right)$



E.g.: $a_1 = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$

$\boxed{a_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz} \iff \int_C f(z)\,dz = 2\pi i\ a_{-1}$

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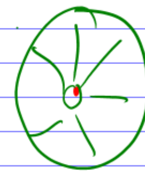
Panel 5

Ex:

$$\int_{|z|=1} \sin(z)\,dz = 0$$

$$\int_{|z|=1} \frac{1}{z^4}\sin(z)\,dz = 2\pi i\left(-\tfrac{1}{6}\right)$$



a_{-1}

$$\frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = \frac{1}{z^3} - \frac{z^{-1}}{3!} + \frac{z^1}{5!} - \dots$$

$$\int_{|z|=1} z^4 \sin(1/z)\,dz = 2\pi i \frac{1}{5!}$$

a_{-1}

$$z^4\left(\frac{1}{z} - \frac{1}{z^3}\frac{1}{3!} + \frac{1}{z^5}\frac{1}{5!} - \dots\right) = z^3 - z/3! + \frac{1}{z}\frac{1}{5!} - \dots$$

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Panel 6

Def: If f is analytic in $0<|z-z_0|<R$ and $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ for $0<|z-z_0|<R$, we define the Residue of f at z_0, or

$$\mathrm{Res}(f, z_0) = a_{-1}$$

Old-fashioned, $z_0 = 0$

$$\int_{|z|=R} f(z) = \int_{|z|=R} \dots + a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + \dots \,dz$$

$$= \dots a_{-3}\int z^{-3}\,dz + a_{-2}\int z^{-2}\,dz + a_{-1}\int z^{-1}\,dz + a_0\int 1 + a_1\int z\,dz + \dots$$

$z = Re^{it}$

$dz = R\,ie^{it}\,dt$

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Panel 7

$$\dots a_{-3}\int z^{-3}\,dz + a_{-2}\int z^{-2}\,dz + a_{-1}\int z^{-1}\,dz + a_0\int 1 + a_1\int z\,dz + \dots$$

$z = Re^{it}$

$dz = Rie^{it}$

$$\dots a_{-3}\int (Re^{it})^{-3}\,Rie^{it}\,dt + a_{-2}\int_0^{2\pi} (Re^{it})^{-2}\,Rie^{it}\,dt +$$

$$a_{-1}\int_0^{2\pi} (Re^{it})^{-1}\,Rie^{it}\,dt + a_0\int_0^{2\pi} 1\,iRe^{it}\,dt + a_1\int_0^{2\pi} Re^{it}\cdot Rie^{it}\,dt + \dots$$

$$= a_{-3}\int_0^{2\pi} iR^{-2}e^{-2it}\,dt + a_{-1}\,2\pi i + \quad = \int_C f(z)\,dz$$

$$= a_{-3}\,iR^{-2}\left(\frac{1}{-2i}\,e^{-2it}\Big|_0^{2\pi}\right) = 0$$

"left-over after integration" = Residue

Panel 8

<u>Ex:</u> Consider $f(z) = \dfrac{-2}{z(z-1)}$. How many residues does f have, and what are they?

$\text{Res}(f, z_0) = a_{-1}$ IF series is centered at z_0, i.e. $(z-z_0)^n$

$$\text{Res}(f,0):\ \frac{1}{z}\cdot\frac{2}{1-z} = \frac{2}{z}\sum_0^\infty z^n = \frac{2}{z}(1+z+z^2\dots) \Rightarrow \underline{\text{Res}(f,0)=2}$$

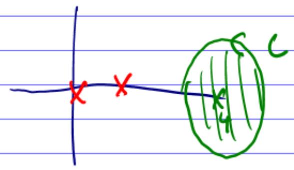
$$\text{Res}(f,1):\ \frac{-2}{z-1}\ \frac{1}{1+(z-1)} = -\frac{2}{z-1}\sum_0^\infty (-1)^n(z-1)^n = -\frac{2}{z-1}\left(1-(z-1)+(z-1)^2-\dots\right)$$

$$\Rightarrow \underline{\text{Res}(f,1) = -2}$$

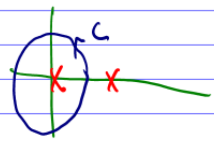
Panel 9

Ex: $\displaystyle\int_C \frac{-2}{z(z-1)}\,dz$ for a) $C: |z-4|=1$

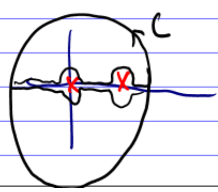
b) $C: |z| = 1/2$ c) $C: |z|=2$

a) 

$$\int_C \frac{-2}{z(z-1)}\,dz = 0$$

b) 

$$\int_C \frac{-2}{z(z-1)}\,dz = 2\pi i \,\mathrm{Res}(f,0)$$

c) 

$$\int_C \frac{-2}{z(z-1)}\,dz = 2\pi i\left(\mathrm{Res}(f,0) + \mathrm{Res}(f,1)\right) = 0$$

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Panel 10

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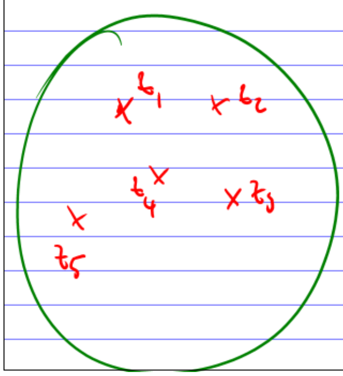
Panel 11

The Residue Theorem

Semester Goal achieved!

Suppose f is analytic in a domain D except for finitely many isolated singularities $z_1, z_2, z_3, \dots, z_n$. If C is a simple curve in D, positively oriented, such that $z_1, z_2, \dots, z_n$ are inside C, then



$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{\infty} \operatorname{Res}(f, z_i)$$

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Panel 12

Ex: $\displaystyle\int_C \frac{5z-2}{z(z-1)}\,dz$ $\{C$ is circle, radius $2\} = 2\pi i(5) = 10\pi i$

$= 2\pi i\left(\operatorname{Res}(f,0) + \operatorname{Res}(f,1)\right)$

$\operatorname{Res}(f,0)$: $\dfrac{5z-2}{z(z-1)} = \dfrac{5z-2}{z} \cdot \dfrac{-1}{1-z} = \left(5 - \dfrac{2}{z}\right)\left(-1 - z - z^2 - z^3 - \dots\right)$

$\operatorname{Res}(f,0) = 2$

$\dfrac{5z-2}{z-1} = \dfrac{5z-5+3}{z-1} = 5 + \dfrac{2}{z-1}$

$\operatorname{Res}(f,1) = \dfrac{5z-2}{z-1} \cdot \dfrac{1}{1+z-1} = \dfrac{5z-2}{z-1} \cdot \sum_0^{\infty} (-1)^n (z-1)^n =$

$\left(5 + \dfrac{3}{z-1}\right)\left(1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots\right)$

$\operatorname{Res}(f,1) = 3$

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Panel 13

Goal: Finding Residues!

First...

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Panel 14

Classification of Singularities:

f has singularity at z_0, i.e. f is not analytic at z_0

$$\Rightarrow \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \ldots a_{-3}(z-z_0)^{-3} + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \ldots$$

case I: all negative powers have zero coefficients
$\Rightarrow$ z_0 is fake, removable singularity

case 2: finitely many neg. powers, N being the largest of them
$\Rightarrow$ z_0 is pole of order N

case 3: inf. many neg. powers
$\Rightarrow$ z_0 is essential singularity

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Panel 15

Ex: $f(z) = \frac{z^2 - 2z + 3}{z-2}$ Sing: $z_0 = 2$ Pole order 1

$g(z) = \frac{1}{z^2(z+1)}$ Sing: $z = 0$ Pole order 2

$z = -1$ Pole order 1

$\frac{1}{z^4}\left(z - \frac{z^3}{3!} + \dots\right) = h(z) = \frac{\sin(z)}{z^4}$ Sing: $z = 0$ Pole order $\cancel{4}$ 3

$f(z) = \frac{z^2 - 9}{z - 3}$ Sing $z = 3$ $\frac{z^2-9}{z-3} = z+3$ Removable

$g(z) = \frac{1 - \cos(z)}{z^2}$ Sing $z = 0$ Pole order $\cancel{2}$ Removable

$\Rightarrow h(z) = e^{1/z}$ Sing. $z = 0$ $\sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$ essential

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Panel 16

HW Find the residues for the functions as given:

a) $\operatorname{Res}(f, 0)$, $f(z) = z^5 \cos(1/z)$

b) $\operatorname{Res}(f, 0)$, $f(z) = z^2 e^{1/z}$

c) $\operatorname{Res}(f, 0)$, $f(z) = \frac{3}{z(z+2)}$

d) $\operatorname{Res}(f, -2)$, $f(z) = \frac{3}{z(z+2)}$

e) $\operatorname{Res}(f, 3)$, $f(z) = \frac{3}{z(z+2)}$

f) $\operatorname{Res}(f, 0)$, $f(z) = \frac{1}{z^3(z-2)}$

HW

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