Panel 1

① Laurent's Theorem

f analytic for $R_1 < |z-z_0| < R_2$. Then

$$f(z) = \dots + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

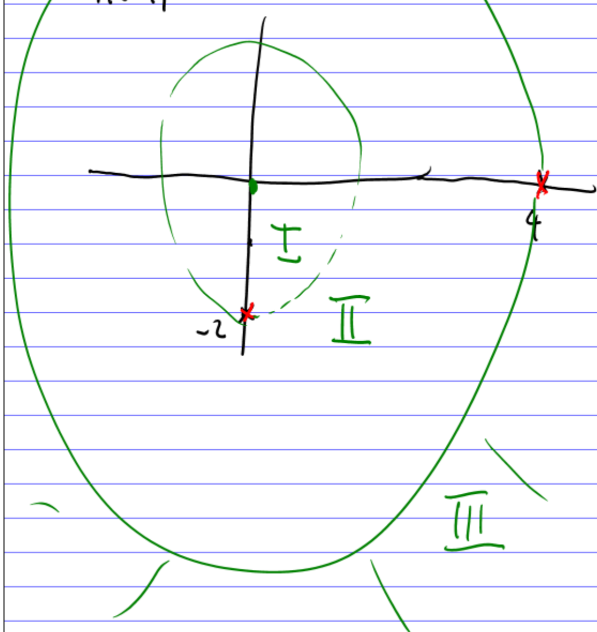
$$= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{for } R_1 < |z-z_0| < R_2$$

$$= \sum_{n=-\infty}^{-1} a_n (z-z_0)^n + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

1

Panel 2

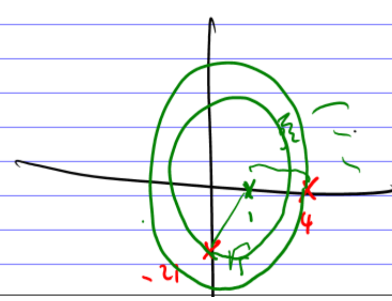
$$\frac{1}{(z+2i)(z-4)} = \sum_{0}^{\infty} a_n (z-1)^n$$



I: $\sum_{0}^{\infty} a_n z^n$

II: $\sum_{-\infty}^{\infty} a_n z^n$

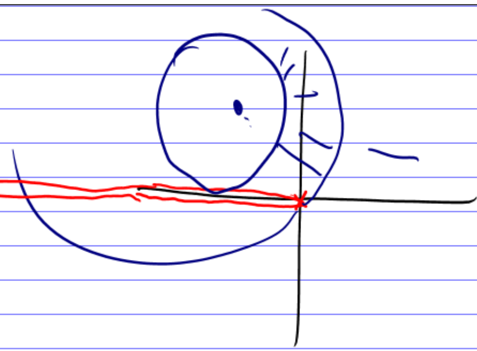
III: $\sum_{n=-1}^{\infty} a_n z^n$

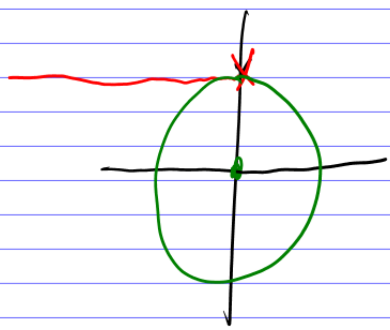


2

Panel 3

$\mathrm{Log}(z-(i-1))$





[d] $\mathrm{Log}(z-i) = \sum_{n=0}^{\infty} a_n z^n$. No series

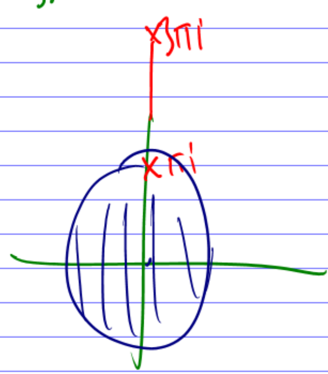
3

Panel 4

$$\frac{1}{e^z-1} = \sum a_n z^n$$

$$\frac{1}{\left(\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots\right)+1\right)} =$$

$$1 \div \left(z+\frac{z^2}{2}+\frac{z^3}{3!}+\dots\right) = \frac{1}{z}$$



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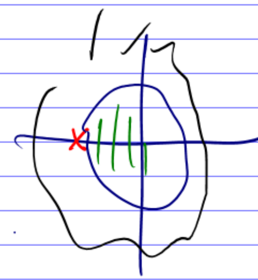
Panel 5

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$z^5 e^{2z} = z^5 \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} z^{n+5}$$

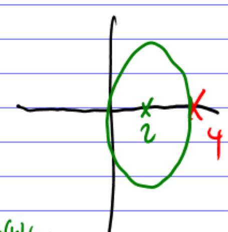
$$\frac{z^3}{1+z^3} = z^3 \frac{1}{1-(-z^3)} = z^3 \sum_{0}^{\infty} (-z^3)^n = \sum_{n=0}^{\infty} (-1)^n z^{3n+3} \Rightarrow |z|<1$$

$$\frac{1}{\frac{1}{z^3}+1} = \sum (-1)^n \frac{1}{z^{3n}} \qquad \left|\frac{1}{z^3}\right|<1 \iff 1<|z|$$



Panel 6

$$\frac{1}{z-4} = \sum a_n (z-2)^n, \quad |z-2|<$$



$f(z) = (z-4)^{-1}, \quad f(2) = -\frac{1}{2}$

$f'(z) = -(z-4)^{-2}, \quad f'(2) = -\frac{1}{4}$

$f''(z) = 2(z-4)^{-3}, \quad f''(2) = -2 \cdot \frac{1}{8}$

$f'''(z) = -3!(z-4)^{-4} \quad f'''(2) = -3!\frac{1}{16}$

$f^{(4)}(z) = 4!(z-4)^{-5}$

$\vdots$

$$a_n = \frac{f^{(n)}(2)}{n!} = -\left(\frac{1}{2}\right)^{n+1}$$

$$\frac{1}{z-4} = -\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} (z-2)^n$$

Panel 7

$$\frac{1}{z-4} = \sum a_n (z-2)^n$$

$$\frac{1}{z-4} = \frac{1}{(z-2)-2} = -\frac{1}{2-(z-2)} = -\frac{1}{2}\frac{1}{1-\left(\frac{z-2}{2}\right)} = -\frac{1}{2}\sum_0^\infty \left(\frac{z-2}{2}\right)^n$$

$$= -\sum_0^\infty \left(\frac{1}{2}\right)^{n+1} (z-2)^n$$

$$e^z = \sum a_n (z-1)^n$$

$$e^{z-1+1} = e^{z-1} e = e\sum_{n=0}^\infty \frac{(z-1)^n}{n!}$$

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Panel 8

$$f(z) = z^4 \sin\left(\frac{1}{z}\right) = z^4 \sum_{n=0}^\infty (-1)^n \frac{\left(\frac{1}{z}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} z^{-(2n+1)}$$

$$= z^4 \left(\left(\frac{1}{z}\right) - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \dots \right) =$$

$$= z^3 - z/3! + \frac{1}{z}\frac{1}{5!} \qquad \sum_{n=-\infty}^{\infty} a_n z^n$$

$$a_2 \text{ for } \left(e^z - 1 - z\right) = \left(1 + z + \frac{z^2}{2} + \dots\right) - 1 - z = \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$$\left(\frac{1}{z}\right)^4 \sin(z) = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right) = \frac{1}{z^3} - \frac{1}{z\,3!}$$

8

Panel 9

$$\frac{1}{3-z} = \begin{cases} \dfrac{1}{3}\cdot\dfrac{1}{1-\frac{z}{3}} = \dfrac{1}{3}\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n = \sum_{n=0}^{\infty}\frac{1}{3^{n+1}}z^n, \quad \underline{|z|<3} \\[2ex] \dfrac{1}{z}\dfrac{1}{\frac{3}{z}-1} = -\dfrac{1}{z}\dfrac{1}{1-\frac{3}{z}} = -\dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n \end{cases}$$

$$\underline{|z|>3}$$

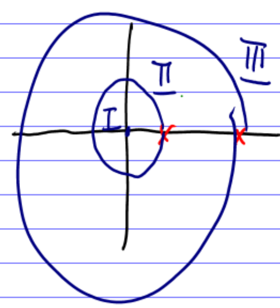
$$= -\sum_{n=0}^{\infty} 3^n/z^{n+1}$$

$$= -\sum_{n=0}^{-\infty} 3^{-n} z^{-n-1}$$

9

Panel 10

$$\frac{-2}{(z-1)(z-3)} = \frac{1}{z-1} - \frac{1}{z-3}$$



$$\underline{\mathrm{I}}: \quad -\frac{1}{1-z} + \frac{1}{3-z} =$$

$$-\sum_{n=0}^{\infty} z^n + \frac{1}{3}\sum_{n=0}^{\infty}\frac{z^n}{3^n} = \sum_{n=0}^{\infty}\left(\frac{1}{3^{n+1}}-1\right)z^n, \quad |z|<1$$

$$\Uparrow \qquad\qquad \Uparrow$$

$$|z|<1 \qquad\qquad |z|<3$$

$$\underline{\mathrm{II}}: \quad \frac{1}{z-1} + \frac{1}{3-z} = \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} + \frac{1}{3}\sum_{0}^{\infty}\frac{z^n}{3^n} = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{3^n}, \quad \underline{\underline{1<|z|<3}}$$

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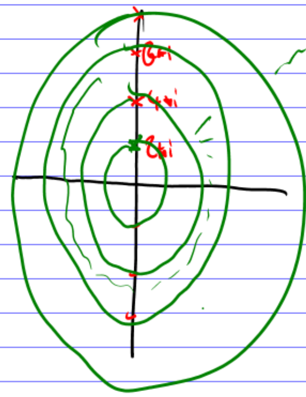
Panel 11

$f(z) = \frac{1}{e^z - 1}$



has inf. many Laurent series.

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