Panel 1

Last Time:

Power series: $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ centered at z_0

Taylor's Thm: $f(z)$ analytic in $|z-z_0|<R$. Then

$$f(z) = \sum_{0}^{\infty} a_n (z-z_0)^n \quad |z-z_0|<R$$

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

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Panel 2

Special Power Series every educated person must know:

$$f(z) = \frac{1}{1-z} = \sum_{0}^{\infty} z^n$$

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$f(z) = \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$f(z) = \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z)^{2n+1}}{(2n+1)!}$$

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Panel 3

Find a series expansion for

$0 < |z| < 1$

$$f(z) = \frac{1}{z^3 - z^4}$$

centered at $z_0 = 0$

$$= \frac{1}{z^3}\frac{1}{1-z} = \frac{1}{z^3}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-3} = \underbrace{\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z}}_{\text{new}} + \overline{1 + z + z^2 + \cdots}$$

"Power series" with neg. exponents

$$f(z) = z e^{z^2} = z\left(\sum_{n=0}^{\infty} \frac{(z^2)^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{n!}$$

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Panel 4

Radius of Convergence Theorem:

① If $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a power series, then it converges for $|z - z_0| < R$ where $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

② If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a Taylor series, then it converges for $|z - z_0| < R$ where R is largest radius where f is analytic.

Ex ① $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ $a_n = 1/n!$

$$R = \lim_{n\to\infty} \frac{1/n!}{1/(n+1)!} = \frac{(n+1)!}{n!} = \frac{(n+1)n!}{n!} = n+1 \to \infty$$

$\sum z^n = \frac{1}{1-z}$ $\forall |z| < 1$

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Panel 5

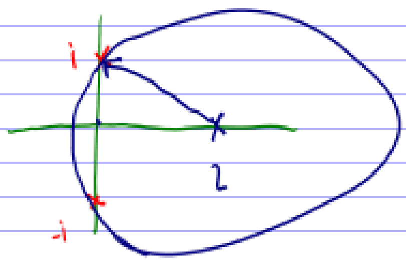
Ex: If $\frac{1}{2-z} = \sum_{n=0}^{\infty} a_n z^n$ for $|z|<R$, find R

$\Leftarrow |z|<2$ $\qquad$ $R=2$

Ex: If $\frac{1}{1+z^2} = \sum_{n=0}^{\infty} a_n (z-2)^n$ for $|z-2|<R$, find R

$R=\sqrt{5}$ $\qquad$ $f = (1+z^2)^{-1}$

$f' = \frac{-2z}{(1+z^2)^2}$



$a_n = \frac{f^{(n)}(2)}{n!}$ e.g.

$f'' = \frac{-2(1+z^2)^2 + 2z \cdot 2(1+z^2) 2z}{(1+z^2)^4}$

$a_2 = \frac{f''(2)}{2!}$

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Panel 6

If f is analytic at z_0 $\Rightarrow$ f has Taylor Series

converges for $|z-z_0|<R$

If f is analytic in $0<|z-z_0|<R$

can not write $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

because series can not converge at $z=z_0$

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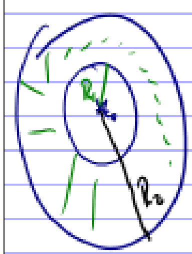
Panel 7

Laurent Theorem: If f is analytic in a ring-shaped domain $R_1 < |z - z_0| < R$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z-z_0)^n}, \quad R_1 < |z-z_0| < R_2$$



where $$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad \left(= \frac{f^{(n)}(z_0)}{n!} \right)$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$$

e.g. $$b_2 = \frac{1}{2\pi i} \int \frac{f(z)}{(z-z_0)^{-1}} dz = \frac{1}{2\pi i} \int_C f(z)(z-z_0)\, dz \neq 0$$

Panel 8

In short:

f analytic on $|z - z_0| < R$, then:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

f analytic in $R_1 < |z-z_0| < R_2$, then:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \dots \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

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Panel 9

Ex: Find Laurent series for

$f(z) = e^{1/z}$ centered at $z_0 = 0$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = \sum_{n=0}^{\infty} z^{-n} \frac{1}{n!} = \sum_{n=-\infty}^{0} z^{n} \frac{1}{(-n)!} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \ldots$$

$g(z) = \frac{1}{(z-i)^2}$ centered at $z_0 = i$

$$= (z-i)^{-2}$$

$h(z) = z^5$ at $z=0$ Taylor in z^5

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Panel 10

Some more Laurent Series centered at zero

$$g(z) = z^3 \cos\left(\frac{1}{3z}\right) = z^3 \sum_{n=0}^{\infty} (-1)^n \frac{(1/3z)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n}(2n)!} \frac{1}{z^{2n-3}}$$

$$= z^3 \left(1 - \frac{\left(\frac{1}{3z}\right)^2}{2!} + \frac{\left(\frac{1}{3z}\right)^4}{4!} - \frac{\left(\frac{1}{3z}\right)^6}{6!} + \ldots\right) =$$

$$= z^3 - \frac{1}{9 \cdot 2!} z + \frac{1}{3^4 4!} z^{-1} - \ldots$$

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Panel 11

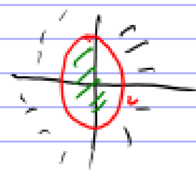
Ex: Take $f(z) = \frac{1}{2-z}$. Find series centered at 0

a) that converges for $|z| < 2$

$$\frac{1}{2-z} = \frac{1}{2\left(1-\frac{z}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n \quad , \left|\frac{z}{2}\right| < 1 \Rightarrow \underline{\underline{|z| < 2}}$$

b) that converges for $|z| > 2$

$$\frac{1}{2-z} = \frac{1}{-z\left(1-\frac{2}{z}\right)} = \frac{1}{-z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} \quad , \left|\frac{2}{z}\right| < 1 \Rightarrow \underline{\underline{|z| > 2}}$$



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Panel 12

Ex: $f(z) = \frac{1}{3-z}$. Series centered at $z_0 = 1$ s.t.

a) convergent for $|z-1| <$

HW

b) convergent for $2 < |z-1|$

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Panel 13

Consider $f(z) = \dfrac{-1}{(z-1)(z-2)}$

Write $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$

3 series!

Hint: PFD

HW