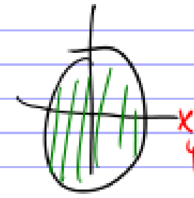
Panel 1

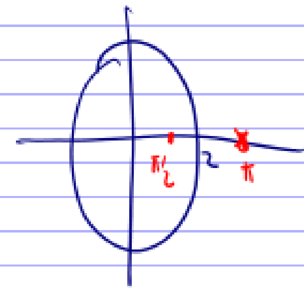
$$\oint_C \frac{z e^{e^z}}{(z-4)^2}\, dz = 0, \quad |z| = 1$$

analytic, so Cauchy Goursat

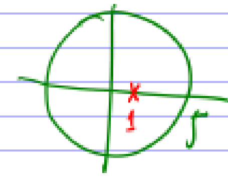


$$\oint \frac{z^2 \cos(z)}{(z-\pi)^2 (z-\pi/2)}\, dz, \quad |z| = 2$$

$$= \int \frac{f(z)}{(z-\pi/2)}\, dz = 2\pi i\, f(\pi/2) = 0$$



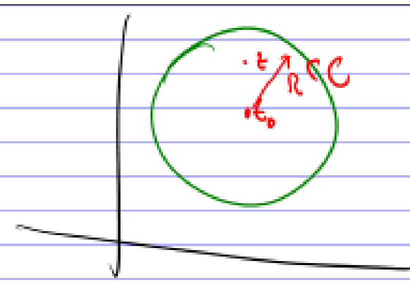
$$\oint \frac{z^3 - 2z^2 + 1}{(z-1)^5}\, dz = \int \frac{f(z)}{(z-1)^5}\, dz = \frac{2\pi i}{4!} f^{(4)}(1) = 0$$



Panel 2

Last Time:

Power series: $\sum_{n=0}^{\infty} a_n (z-z_0)^n$



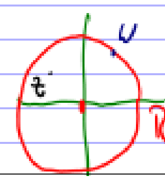
Taylor's Thm: Let f be analytic in $|z-z_0| < R$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!}, \quad |z-z_0| < R.$$

Proof: Know

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw \quad \forall z \text{ inside } |z-z_0| < R$$

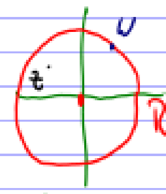
For simplicity, take $z_0 = 0$.



Panel 3

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw \qquad \forall z \text{ inside } |z|<R$$

$$= \frac{1}{2\pi i}\int_C \frac{f(w)}{w\left(1-\frac{z}{w}\right)}\,dw$$



$$\left|\frac{z}{w}\right|<1$$

$$= \frac{1}{2\pi i}\int_C \frac{f(w)}{w}\cdot\frac{1}{1-\frac{z}{w}}\,dw = \frac{1}{2\pi i}\int \frac{f(w)}{w}\sum\left(\frac{z}{w}\right)^n dw =$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int \frac{f(w)}{w^{n+1}}\,dw\right] z^n =$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{n!}f^{(n)}(0)\right] z^n = \sum_{n=0}^{\infty} a_n z^n,$$

$$a_n = \frac{1}{n!}f^{(n)}(0)$$

\#

3

Panel 4

$f(z) = \cosh(z)$	$f(0)=1$
$f'(z) = \sinh(z)$	$f'(0)=0$
$f''(z) = \cosh(z)$	$f''(0)=1$
$f'''(z) = \sinh(z)$	$f'''(0)=0$
$\vdots$	

$$\Rightarrow f(z) = a_0 + \cancel{\frac{a_1}{1!}z} + \frac{a_2}{2!}z^2 + \cancel{\frac{a_3}{3!}z^3} + \dots$$

$$= 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \dots$$

$$= \sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}$$

<u>Method 1:</u> find $f', f'', f^{(4)}, \dots$, look for pattern

4

Panel 5

$f(z) = z^3 e^{(2z)}$ Method I is no good because f' are complicated.

Method II: use known series!

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow e^{(2z)} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!}$$

$$\Rightarrow z^3 e^{(2z)} = \sum_{n=0}^{\infty} \frac{2^n z^{n+3}}{n!}$$

$$\lim_{z \to 0} \frac{e^z - 1 - z}{z^2} = \lim_{z \to 0} \frac{\left(\cancel{1} + \cancel{z} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right) - \cancel{1} - \cancel{z}}{\cancel{z^2}} = \lim_{z \to 0}\left(\frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots\right) = \frac{1}{2}$$

Panel 6

Special Power Series every educated person must know:

$$f(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \dots = \sum_{n=0}^{\infty} z^n \quad, |z| < 1$$

$$f(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \forall z$$

$$f(z) = \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$f(z) = \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$e^{it} = 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} \dots \quad \cos(t) + i\sin(t)$$

Panel 7

Find McLaurin Series for $f(z) = 1/_{1-z}$, $g(z) = z/_{z^2+4}$

$g(z) = z \frac{1}{1-}$

$\frac{z}{z^2+4} = z \frac{1}{z^2+4} = z \frac{1}{4\left(1+\frac{z^2}{4}\right)} =$

$= \frac{z}{4} \cdot \frac{1}{1-\left(-\left(\frac{z}{2}\right)^2\right)} =$

$= \frac{z}{4} \sum_{n=0}^{\infty} \left(-\left(\frac{z}{2}\right)^2\right)^n = \frac{z}{4} \sum (-1)^n \frac{z^{2n}}{4^n} =$

$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{4^{n+1}}$

7

Panel 8

Find McLaurin Series for $f(z) = z e^{z^2}$

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow z e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{n!}$

Prove that.

$\lim_{z \to 0} \frac{\sin(z)}{z} = \lim_{z \to 0} \frac{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots\right)}{z} = \lim_{z \to 0} \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots\right) = 1$

$\lim_{z \to 0} \frac{\cos(z) - 1}{z} = \lim_{z \to 0} \frac{\left(\cancel{1} - \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right) - \cancel{1}}{z} = 0$

8

Panel 9

Find power series for $f(z) = 1/z$ centered at $c = \cancel{0}\ 3$

Want: $\sum a_n (z-3)^n$ $\qquad |z-3| < 3$

$$\frac{1}{z} = \frac{1}{3 + (z-3)} = \frac{1}{3} \frac{1}{1 - \frac{z-3}{(-3)}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{z-3}{-3}\right)^n \qquad , \left|\frac{z-3}{-3}\right| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{n+1}} (z-3)^n$$

9

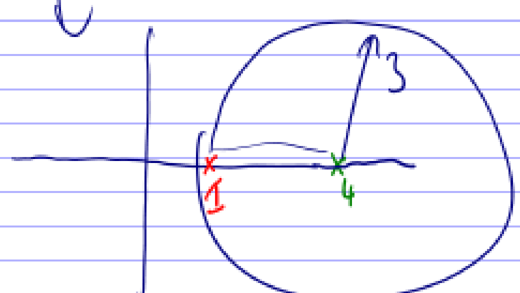
Panel 10

Find a series expansion for

$f(z) = \frac{1}{1-z}$ centered at 4.

$$= \frac{1}{-3 - (z-4)} = \frac{1}{-3} \frac{1}{1 - \frac{z-4}{-3}} = -\sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{n+1}} (z-4)^n$$

$$\left|\frac{z-4}{3}\right| < 1 \Leftrightarrow \underline{|z-4| < 3}$$



10

Panel 11

Radius of Convergence Theorem:

(1) If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is a power series, then it converges for $|z-z_0|<R$ where $R = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

(2) If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is a Taylor series, then it converges for $|z-z_0|<R$ where R is the largest radius for which f is analytic in $|z-z_0|<R$

11