Panel 1

Foundational Theorems of Complex Analysis

1. Cauchy-Riemann Equations
2. Cauchy-Goursat Thm
3. Deformation thm
4. Path Independence thm
5. Cauchy Integral Formula
6. General Cauchy Int. Formula
7. Morera's thm
8. Cauchy's Inequality
9. Liouville's thm
10. Fund. thm. of Algebra
11. Max Mod. Principle

Pick your three favorite thms and write them down. (Incl. your name!)

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Panel 2

Consequence of Fund. Thm. of Algebra: If $p_n(z)$ is a complex polynomial of degree n then there are n solutions $p_n(z_i) = 0$, $i = 0, 1, \dots, n-1$

Know $p_n(z) = 0$ for at least one z_0

$\Rightarrow \dfrac{p_n(z)}{z - z_0} = p_{n-1}(z)$, degree $n-1$

$\Rightarrow p_{n-1}(z_1) = 0$ for at least one z_1

$\Rightarrow \dfrac{p_{n-1}(z)}{z - z_1} = p_{n-2}(z)$ $\rightsquigarrow$ $z_0, z_1, \dots, z_{n-1}$ solutions!

q.e.d.

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Panel 3

Some Brain Gymnastics:

① If f is entire and $|f(z)| \le M|z|$ then $f(z) = cz$

② Let $$g(z) = \int_{|s|=3} \frac{2s^2 - s - 2}{s - z}\, ds, \quad |z| \ne 3$$

Find $g(2)$ and $g(5)$

③ Let C be the circle $|z-i| = 2$. Find

a) $\int_C \frac{1}{z^2+4}\, dz$ b) $\int_C \frac{1}{(z^2+4)^2}\, dz$

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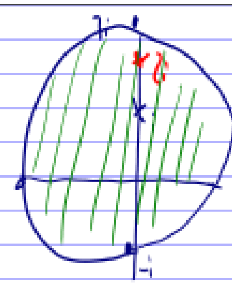
Panel 4

③ Let C be the circle $|z-i| = 2$. Find



$$\int_C \frac{1}{z^2+4}\, dz = \int_C \frac{\overbrace{\frac{1}{z+2i}}^{f(z)}}{z-2i}\, dz =$$

$$\int_C \frac{f(z)}{z-2i}\, dz = 2\pi i\, f(2i) = 2\pi i \frac{1}{4i} = \frac{\pi}{2}$$

$z_0 = 2i$

b) $$\int_C \frac{1}{(z^2+4)^2}\, dz = \int_C \frac{\overbrace{\frac{1}{(z+2i)^2}}^{f(z)}}{(z-2i)^2}\, dz = \int_C \frac{f(z)}{(z-2i)^2}\, dz =$$

$f(z) = (z+2i)^{-2}$

$$= \frac{2\pi i}{1!} f'(2i) = \frac{2\pi i}{1}\left(-2(4i)^{-3}\right) = \frac{-4\pi i}{-4^3 i} = \frac{\pi}{16}$$

$f'(z) = -2(z+2i)^{-3}$

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Panel 5

② Let $g(z) = \int_{|s|=3} \frac{2s^2 - s - 2}{s - z}\, ds \quad , |z| \neq 3$

$g(5) = \int_{|s|=3} \frac{2s^2-s-2}{s-5}\, ds = 0$, Cauchy - Goursat

$$g(2) = \int_{|s|=3} \frac{2s^2-s-2}{s-2}\, ds = 2\pi i\, f(2) = 2\pi i\,(4) = 8\pi i$$

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Panel 6

① If f is entire and $|f(z)| \le M|z|$ then $f(z) = cz$

Louiville's Thm: $|f(z)| \le M$, f entire $\Rightarrow |f'(z)| \le \frac{1!M}{R} \to 0$ $\forall R$

$f' = 0 \Rightarrow f$ const

Idea: $|f''(z)| = 0$ using Cauchy's Ineqn.

If $|f(z)| \le K$ $|z - z_0| < R \Rightarrow |f^{(n)}(z_0)| \le \frac{n!K}{R^n}$

Pick $|z - z_0| < R$. Know

$|z| - |z_0| \le |z - z_0| < R \Rightarrow |z| \le R + |z_0|$

$\Rightarrow |f(z)| \le M \cdot |z| \le M(R + |z_0|) \Rightarrow |f''(z_0)| \le \frac{M(R+|z_0|)\cdot 2}{R^2} \to 0$ as $R \to \infty$

$|f''(z)| = 0 \Rightarrow f(z) = cz + \cancel{b}$ but $|f(z)| \le c|z| \Rightarrow f(0) = 0 \Rightarrow b = 0$

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Panel 7

There is one more important Theorem, called the Maximum Modulus Principle.

Recall: If f is a function of 2 real variables on a closed, bdd set in $\mathbb{R}^2$ $\Rightarrow$ f has max + min. Moreover, they can occur at critical points or at the boundary.

X ① $\nabla f = (f_x, f_y) = 0$

② On bdry of D.

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Panel 8

Maximum Modulus Principle: If f is analytic in a domain D and not constant, then $|f(z)|$ has no max. inside D [or alt.: any max. must occur on bdry]

Corollary: f analytic inside D and cont. on bdry of D, then f has a max on the bdry.

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Panel 9

Ex: Find abs. max. of $|f(z)|$, where $f(z) = z^2$, over the unit disk.

$|f(z)| = |z^2| = |z|^2 = x^2+y^2$.

$\nabla f = 0 \Leftrightarrow (2x, 2y) = (0,0) \Rightarrow x=0, y=0$ are critical

$Hf = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0,\ f_{xx} > 0 \Rightarrow$ critical is min.

On $\partial D = x^2+y^2=1 \Rightarrow |f(z)| = 1$ is .

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Panel 10

Complex Analysis — High lights

1. Algebra in $\mathbb{C}$ — Euler's formula
2. Derivatives in $\mathbb{C}$ — CR-equations
3. Integrals in $\mathbb{C}$ — (General) Cauchy Int. Formula

Next: Infinite Series.

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Panel 11

Infinite Series: The formal sum

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + z_3 + z_4 + \dots + z_n + \dots$$

is called infinite series, or just Series.

The series $\sum_{n=0}^{\infty} z_n$ converges if the sequence

$S_N = z_0 + z_1 + \dots + z_N$ of partial sums

converges as a sequence.

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Panel 12

Ex: Find out where the sum of $\sum_{n=0}^{\infty} 2^n$ converges.

$S_0 = 2^0 = 1$

$S_1 = 2^0 + 2^1 = 3$

$S_2 = 2^0 + 2^1 + 2^2 = 7$

$S_3 = 2^0 + 2^1 + 2^2 + 2^3 = 15$

S_4

31

63

Sequence of partial sums.

Does not converge!

S) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ converges to 2

Surprise!

$S_0 = \left(\frac{1}{2}\right)^0 = 1$

$S_1 = \left(\frac{1}{2}\right)^0 + \frac{1}{2} = \frac{3}{2}$

$S_2 = \left(\frac{1}{2}\right)^0 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$

$S_3 = \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right) + \frac{1}{4} + \frac{1}{8} = \frac{15}{8} \sim$

$\frac{31}{16} \sim 2$

$\frac{63}{32} \sim 2$

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Panel 13

Ex: $\sum_{n=0}^{\infty} z^n$ for any z. ($z=2 \Rightarrow$ div, $z=1/2 \Rightarrow$ conv.)

$S_0 = 1$

$S_1 = 1+z$

$S_2 = 1+z+z^2$

$\vdots$

$S_n = 1+z+z^2+\dots+z^n$

$z \cdot S_n = z+z^2+z^3+\dots+z^{n+1}$ —

$S_n - zS_n = 1 - z^{n+1}$

$\Rightarrow S_n(1-z) = 1 - z^{n+1}$

$S_n = \frac{1-z^{n+1}}{1-z}$

S_n converges if $|z|^{n+1}$ converges

i.e. if $|z|<1$

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Panel 14

Thm: The Geometric Series $\sum_{n=0}^{\infty} z^n$ converges if $|z|<1$ and diverges if $|z|>1$.

In fact: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ if $|z|<1$

Ex: $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-1/2} = 1/_{1/2} = 2$

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Panel 15

Side bar:

Convert 0.36363636... to a fraction:

$$0.363636\ldots = \frac{36}{100} + \frac{36}{10000} + \frac{36}{1000000} + \ldots$$

$$= 36\left(\frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} + \frac{1}{100^4} + \ldots\right)$$

$$= \frac{36}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} + \ldots\right) =$$

$$= \frac{36}{100}\sum_{k=0}^{\infty}\left(\frac{1}{100}\right)^k = \frac{36}{100}\,\frac{1}{1-\frac{1}{100}} = \frac{36}{100}\cdot\frac{1}{99/100} = \frac{36}{100}\cdot\frac{100}{99} = \frac{36}{99}$$

$$0.415415415\ldots = \frac{415}{999}$$

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