Panel 1

Review of Big Deal Theorems

1. Cauchy - Riemann Eqn.
2. Cauchy - Goursat
3. Deformation Thm
4. Path Independence (Fund. Thm)
5. Cauchy Int. Formula
6. General Cauchy Int. Formula
7. Morera's Thm
8. Cauchy's Inequality
9. Liouville's Thm
10. Fund. Thm. of Alg.

Mon:
3 out of 10
statements only!
Quiz!

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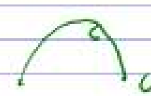
Panel 2

$$\left|\int_C f(z)\,dz\right| \le M\,\text{length}(C) \quad , \; |f(z)| \le M \quad \forall z \in C$$

$$\left|\int_C \frac{z}{z^2-1}\,dz\right| \le M\cdot \text{length}(C) = M\,2\pi = \frac{4\pi}{3}$$



$$\left|\frac{z}{z^2-1}\right| = \frac{|z|}{|z^2-1|} = \frac{2}{|z^2-1|} \le \frac{2}{3} \qquad z \in C,\ |z|=2$$

$$|z^2-1| \ge |z^2| - 1 = 3$$

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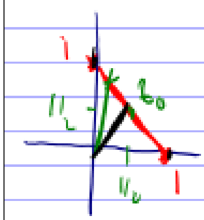
Panel 3

$\left|\int \frac{1}{z^4}\, dz\right| \le 4\sqrt{2}$ ✓ from i to 1 : $r(t) = i + t(1-i)$, $t \in [0,1]$

middle $r\left(\frac{1}{2}\right) = i + \frac{1}{2}(1-i) = \frac{i}{2} + \frac{1}{2}$



$|z| \ge |z_0| = \left|\frac{1}{2} + \frac{1}{2}i\right| = \sqrt{\frac{1}{2}}$

$|z|^4 \ge \left(\sqrt{\frac{1}{2}}\right)^4 = \frac{1}{4} \Rightarrow \frac{1}{|z|^4} \le 4 = M$

$\left|\int \frac{1}{z^4}\, dz\right| \le M \cdot \text{length}(C) = \underline{\underline{4\sqrt{2}}}$ ✓

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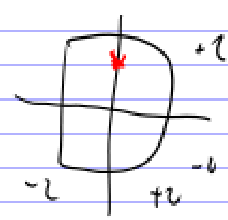
Panel 4

$\int_C \frac{e^{-z}}{z - \frac{\pi i}{2}}\, dz$ $f(z)$ $\rightarrow z_0$



$\int_C \frac{f(z)}{z - z_0} = \underline{2\pi i}\, f(z_0) = 2\pi i \underbrace{\left(e^{-\frac{\pi i}{2}}\right)}_{= -i} = 2\pi$

$\int_C \frac{z}{2z+1}\, dz = \int \frac{\frac{z}{2}}{z - (-\frac{1}{2})}\, dz$ ($f(z)$, z_0)

$\hookrightarrow = f(z_0)\, 2\pi i = \frac{(-1/2)}{2}(2\pi i) = \underline{\underline{\frac{-\pi i}{2}}}$

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Panel 5

$$\int \frac{\cosh(z)}{z^4}\,dz = \qquad \frac{n!}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}}\,dz = f^{(n)}(z_0)$$

$$\int \frac{\overbrace{\cosh(z)}^{f(z)}}{(\underset{z_0}{z-0})^4}\quad n=3 \quad = \frac{2\pi i}{3!} f^{(3)}(0) \quad , \quad f(z)=\cosh(z)$$

$$f'(z) = \sinh(z)$$

$$= \frac{2\pi i}{3!}\sinh(0) = 0 \qquad f''(z) = \cosh(z)$$

$$f'''(z) = \sinh(z)$$

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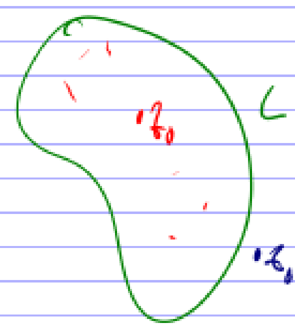
Panel 6

$$g(z) = \int_C \frac{\overbrace{s^3+2s}^{f(s)}}{(s-\underset{z_0}{z})^3}\,ds \quad n=2$$



(a) $z \notin C$: $g(z) = 0$

(b) $z \in C$: $\frac{2\pi i}{2!} f''(z) = \frac{2\pi i}{2!}\, 6z = 6\pi i z$

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Panel 7

Consequences of (General) Cauchy's Int. Formula

① f analytic ⇒ f' is analytic ⇒ f'' analytic ⇒

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^2}\, d\xi$$ every derivative exists

(contrast with $f(x) = x^{4/3}$)

② Corollary: f is analytic ⇒ $u, v \in C^\infty$, i.e. inf. often diff'ble.

$C^n(D) = \{ f : D \to \mathbb{R}_{\mathbb{C}} : f$ is n-times diff'ble, $f^{(n)}$ is cont $\}$

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Panel 8

⑦ Morera's Thm: If D is a domain and

$\int_C f(z)\, dz = 0$ ∀ closed curves in D

then f is analytic.

⇒ ∃ F antiderivative

⇒ $F' = f$, but f must be analytic

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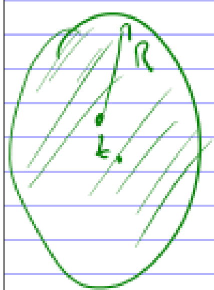
Panel 9

9 Cauchy's Inequality: If f is analytic inside a circle C_R centered at z_0, and $|f(z)| \leq M$ on C_R, then

$$|f^{(n)}(z_0)| \leq \frac{M n!}{R^n}$$



Proof: $$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz \right| \leq$$

$$\frac{n!}{2\pi} K \cdot \text{length}(C) = \frac{n!}{\cancel{2\pi}} \frac{M}{R^{n+1}} \cdot \cancel{2\pi} \cancel{R}$$

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Panel 10

Review:

Cauchy-Goursat Theorem:

Cauchy's Int. Formula:

Cauchy's General Int. Formula:

Cauchy's Inequality:

Morera's Theorem.

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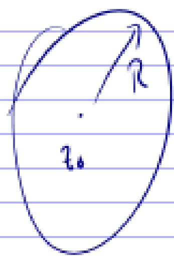
Panel 11

Liouville's Theorem: If f is entire and bounded, then f must be const!

Proof Know $|f(z)| \le M$ for some M.

Pick any z_0, any R $\Rightarrow$ f is analytic in $|z-z_0|<R$

$\Rightarrow |f^{(n)}| \le \frac{n!M}{R^n}$. Pick $n=1$



$\Rightarrow |f'(z_0)| \le \frac{M}{R}$ $\forall R$ $\Rightarrow |f'(z_0)| = 0$

$f'(z_0) = 0$ $\forall z_0$ $\Rightarrow f'(z) \equiv 0 \Rightarrow f$ const. ∎

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Panel 12

Fundamental Theorem of Algebra: If $p(z)$ is a polynomial of degree $n \ge 1$, then $p(z)=0$ for at least one $z_0 \in \mathbb{C}$.

Proof. Take $p(z) = \underline{a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n}$, $a_n \ne 0$

Suppose $p(z) \ne 0$. Then $1/p(z)$ is entire!

$$\frac{1}{p(z)} = \frac{1}{z^n \underbrace{\left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)}}$$

$$\lim_{z \to \infty} a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} = a_n \ne 0$$

If $|z| > R$ $\Rightarrow \left|a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n}\right| \ge |a_n| - 1$ any # would do for some R

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Panel 13

$$\left|\frac{1}{p(z)}\right| = \left|\frac{1}{z^n\left(a_n + \dots + \frac{a_0}{z^n}\right)}\right| \le \frac{1}{|z|^n(|a_n|-1)} \le \frac{1}{R^n(|a_n|-1)}, \quad \underline{|z| \ge R}$$

If $|z| \le R$ then $\frac{1}{p(z)}$ is cont. on a closed, bounded set.

$\Rightarrow$ If $|z| \le R \Rightarrow 1/p(z)$ has max and min, say $\left|\frac{1}{p(z)}\right| \le K$

Let $M = \max\left(K, \frac{1}{R^n(|a_n|-1)}\right)$. Then $\left|\frac{1}{p(z)}\right| \le M \quad \forall z$

$\Rightarrow 1/p(z)$ is entire + bdd $\underset{LV}{\Rightarrow}$ $1/p(z) =$ const $\Rightarrow p(z) = c \Rightarrow$ degree $n = 0$

↯ Contradiction $\Rightarrow p(z) = 0$

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