Panel 1

Integration Theorems

Integral Estimation Theorem: f cont. on curve C and $|f(z)| \le M$ on C. Then

$$\left|\int_C f(z)\,dz\right| \le M \cdot \text{length(curve)}$$

How-To Theorem: f continuous in domain D. Then the following are equivalent:

a) f has antiderivative

b) $\int_\gamma f(z)\,dz = F(z_2) - F(z_1)$ for all γ from z_1 to z_2

c) $\oint_\gamma f(z)\,dz = 0$ $\forall$ closed curves γ.

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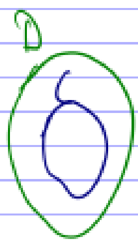
Panel 2

Cauchy-Goursat Theorem: If f is analytic in a simply connected domain D and f is analytic in D then



$$\int_C f(z)\,dz = 0 \qquad \forall\, C \text{ closed curve in } D$$

Deformation Theorem: If C_1 and C_2 are two simple closed curves, positively oriented, with C_1 inside C_2, and f analytic between them:

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

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Panel 3

$\frac{d}{dz} \frac{1}{n+1} z^{n+1} = z^n \Rightarrow F(z) = \frac{1}{n+1} z^{n+1}$ is antideriv of z^n $\forall z$.

$\Rightarrow \int_\gamma z^n = F(z_2) - F(z_1) \quad \forall \gamma$ from z_1 to z_2

$\int_i^{2+i} \bar{z}\, dz$ is illegal - might depend on γ from i to $2+i$

$$\int_i^{i/2} e^{\pi z}\, dz = \frac{1}{\pi} e^{\pi z} \Big|_i^{i/2} = \frac{1}{\pi}\left[e^{\pi i/2} - e^{\pi i}\right] = \frac{1}{\pi}\left[i + 1\right]$$

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Panel 4

$\int \cos(z^2) e^z\, dz$. Clearly $f(z) = \cos(z^2) e^z$ has antideriv. (use cst. path)

use Cauchy Goursat $\Rightarrow \oint_C \cos(z^2) e^z\, dz = 0$

$u = \cos(z^2)$ $\quad v' = e^z$

$u' = -2z \sin(z^2)$ $\quad v = e^z$

$\int \cos(z^2) e^z\, dz = e^z \cos(z^2) + 2 \int z \sin(z^2) e^z$

$u = e^z$ $\quad v' = \cos(z^2)$

$u' = e^z$ $\quad v$

$\int e^{x^2} dx$

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Panel 5

$$\int e^{z^2}\,dz = \int \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}\,dz \overset{?}{=} \sum_{n=0}^{\infty} \int \frac{z^{2n}}{n!}\,dz =$$

$$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)\,n!} + C$$

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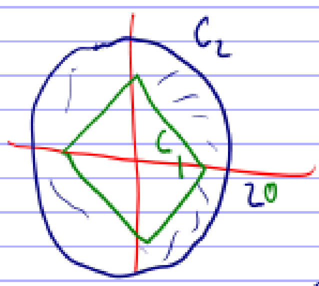
Panel 6



$\int_C \frac{3}{z}\,dz$ & $\frac{3}{z}$ has no antider.

analytic along curve!

$$\int_C \frac{3}{z}\,dz = \int_{C_2} \frac{3}{z}\,dz = \int_0^{2\pi} \frac{3}{\cancel{20e^{it}}} \cdot \cancel{20} i e^{it}\,dt = \underline{3 \cdot 2\pi i}$$

$$z = 20e^{it}, \quad t \in [0, 2\pi]$$

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Panel 7

We proved the "How to" theorem (one part was tricky)
We proved the Cauchy-Goursat thm (used Green's thm)

Integral Estimation: If f is continuous along a curve γ then and $|f(z)| \le M$ on curve γ then:

$$\left|\int_\gamma f(z)\,dz\right| \le$$

Proof:

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right| \le \int_a^b |f(z(t))|\,|z'(t)|\,dt$$

$$\le M\int_a^b |z'(t)|\,dt = M\int \sqrt{(x')^2+(y')^2}\,dt = M\cdot L$$

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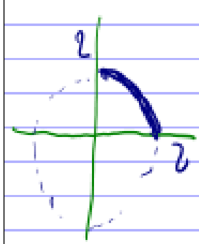
Panel 8

Estimate $\displaystyle\int_\gamma \frac{z+4}{z^3-1}\,dz$, γ is first $\frac{1}{4}$ of $|z|=2$



Estimation thm requires $\left|\frac{z+4}{z^3-1}\right| \le M$ for some # M. Also, length(curve) $= \pi$

Know z are on $|z|=2$ $\Rightarrow$ $|z+4| \le |z|+4 = 6$

$|z^3-1| \ge |z|^3-1 = 7$

$\Rightarrow \left|\frac{z+4}{z^3-1}\right| \le \frac{6}{7}$, for $|z|=2$ $\Rightarrow$ $\left|\int \frac{z+4}{z^3-1}\,dz\right| \le \frac{6}{7}\cdot\pi$

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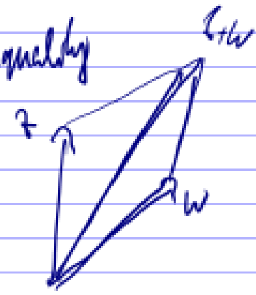
Panel 9

Basic Inequalities:

$|z+w| \leq |z| + |w|$ triangle inequality

$|z-w| \geq |z| - |w|$



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Panel 10

Cauchy's Integration Formula:

f analytic inside and on a simple, closed curve C (pos. oriented).

Then $\displaystyle f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \quad \forall z_0 \text{ in } C$

Ex: $\displaystyle \int_{|z|=2} \frac{\overbrace{z^2+3z+1}^{f(z)}}{z - \underbrace{1}_{z_0}}\, dz = 2\pi i f(z_0) = 2\pi i\left(1^2 + 3 \cdot 1 + 1\right) = 10\pi i$

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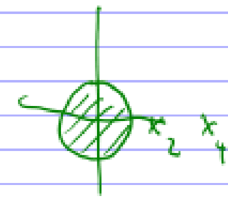
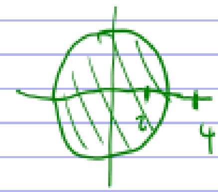
Panel 11

$$\int_{|z|=1} \frac{z}{(z-2)(z-4)}\,dz = 0$$



$$\int_{|z|=3} \frac{\overbrace{z}^{f(z)}}{(z-2)(z-4)}\,dz = \int \frac{f(z)}{z-z_0} =$$



$$= 2\pi i\, f(2) = 2\pi i \frac{2}{-2} = \underline{\underline{-2\pi i}}$$

$$\int_{|z|=5} \frac{z}{(z-2)(z-4)}\,dz$$

$$\frac{1}{2\pi i}\int \frac{f(z)}{z-z_0}\,dz = f(z_0)$$

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Panel 12

$$\int_{|z|=5} \frac{z}{(z-2)(z-4)}\,dz =$$

$$\int_{|z|=5} \frac{-1}{z-2}\,dz + \int_{|z|=5} \frac{2}{z-4}\,dz$$

$$-2\pi i + 2\cdot 2\pi i = \underline{\underline{2\pi i}}$$

$$\frac{z}{(z-2)(z-4)} = \frac{A}{z-2} + \frac{B}{z-4} =$$

$$= \frac{A(z-4)+B(z-2)}{(z-2)(z-4)} = \frac{z}{(z-2)(z-4)}$$

$$A(z-4)+B(z-2) = z \quad \forall z$$

$\underline{z=2}$: $-2A = 2 \Rightarrow A = -1$

$\underline{z=4}$: $2B = 4 \Rightarrow B = 2$

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Panel 13

Cauchy's Int. Formula

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \quad \forall z_0 \in D$$

Replace z_0 by w: $f(w) = \frac{1}{2\pi i} \int \frac{f(z)}{z-w}\, dz \quad \forall w \in D$

$$\frac{d}{dw} f(w) = \frac{1}{2\pi i} \int \frac{d}{dw} \frac{f(z)}{z-w}\, dz = \frac{1}{2\pi i} \int \frac{f(z)}{(z-w)^2}\, dz = f'(w)$$

$$f''(w) = \frac{2}{2\pi i} \int \frac{f(z)}{(z-w)^3}\, dz \Rightarrow f^{(4)}, f^{(n)}, f^{(n)}, \ldots$$

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Panel 14

General Cauchy Integral Formula: f analytic inside and on a simple, closed curve C (pos. oriented). Then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

Ex: $\displaystyle\int_{|z|=1} \frac{e^{2z}}{z^4}\, dz = \int_{|z|=1} \frac{e^{2z}}{(z-0)^4}\, dz = \frac{2\pi i}{3!} \cdot f^{(3)}(0)$, $f(z) = e^{2z}$

$f^{(3)}(z) = 8e^{2z}$

$n=3$

$$= \frac{2\pi i}{6} \cdot 8$$

$$\int_{|z|=1} \frac{e^{2z}}{(z-2)^4}\, dz = 0$$

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Panel 15

$$\int_{|z|=1/2} \frac{\sin(z)e^z}{(z-1)(z-2)}\,dz = 0 \quad \text{by Cauchy Goursat}$$

$$\int_{|z|=2} \frac{\overbrace{\cos(z)e^z}^{f(z)}}{(z-1)^2(z-4)}\,dz = \frac{2\pi i}{1!}\cdot f'(z)\Big|_{z=1} \quad \text{some work}$$

$$\int_{|z|=2} \frac{e^{z^3}}{(z-1)(z-4)^2}\,dz = 2\pi i\, f(1) = 2\pi i \frac{e^1}{(-3)^2} = \frac{2}{9}\pi i e$$

$$\int_{|z|=5} \frac{\cos(z)\sin(z^2)}{(z-1)(z-4)}\,dz = \text{PDF}$$