Panel 1

Last Topic: Integration

$$\int_\gamma f(z)\,dz = \int f(z(t))\,z'(t)\,dt, \qquad \gamma: z(t),\ t\in[a,b]$$

$$\frac{dz}{dt} = z' \Rightarrow dz = z'\,dt$$

$$\int_\gamma z^2\,dz = \int_0^1 [i+t(1-i)]^2(1-i)\,dt = \qquad \gamma: \text{from } i \text{ to } 1$$

$$= \dots \frac{1}{3}(1+i) \qquad z(t) = i + t(1-i),\ t\in[0,1]$$

$$dz = (1-i)\,dt$$

1

Panel 2

Last Theorem: Suppose f is continuous in a domain D. Then the following are equivalent:

a) f has antideriv. F, i.e. $F' = f$

b) If γ_1, γ_2 are any 2 curves from z_1 to z_2, then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz = F(z_2) - F(z_1)$$

c) If γ is any closed path in D then

$$\oint_\gamma f(z)\,dz = 0$$

2

Panel 3

Ex: $\int_\gamma z^2\,dz$, γ line from i to 1.

Antideriv. $F(z) = \frac{1}{3}z^3 + c$

$$\int_\gamma z^2\,dz = \frac{1}{3}(1)^3 - \frac{1}{3}(i)^3 = \frac{1}{3}(1+i)$$

3

Panel 4

Ex: Evaluate $\int_\gamma \frac{1}{z^2}\,dz$ for $|z| = R$ (starting at R, counter clockwise)

① Old-fashioned: $z(t) = Re^{it}$, $t \in [0, 2\pi)$

$dz = Rie^{it}\,dt$

$$\int_\gamma \frac{1}{z^2}\,dz = \int_0^{2\pi} \frac{1}{(Re^{it})^2} Rie^{it}\,dt = \frac{i}{R}\int_0^{2\pi} e^{-it}\,dt = \frac{i}{R}\left(-\frac{1}{i}\right)e^{-it}\Big|_0^{2\pi} = -\frac{1}{R}\left(e^{-2\pi i} - 1\right)$$

$= 0$

② $\int_\gamma \frac{1}{z^2}\,dz = -z^{-1}\Big|_R^R = 0$ (closed loop)

4

Panel 5

Ex: Evaluate $\int_\gamma \frac{1}{z}\,dz$ for $|z| = R$

$\log(z) = \ln(|z|) + i(\operatorname{Arg}(z) + 2k\pi)$

① $\int_\gamma \frac{1}{z}\,dz = \operatorname{Log}(z)\Big|_R^R = 0$

② $z(t) = Re^{it}, \; dz = iRe^{it}\,dt$

$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{Re^{it}}\, iRe^{it}\,dt = i\int_0^{2\pi} dt = \underline{2\pi i}$

What gives ???!

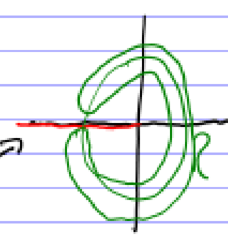
$\frac{d}{dz}\operatorname{Log}(z) = \frac{1}{z}$ (with restrictions)

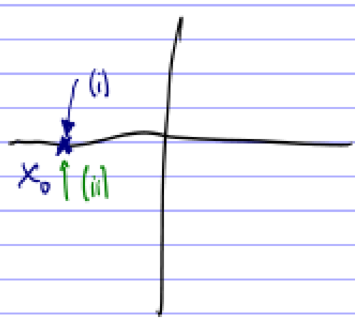
5

Panel 6

Thm: $\operatorname{Log}(z)$ is not analytic for $\{\operatorname{Re}(z) \le 0$ $\operatorname{Im}(z) = 0$ and $\}$

In fact, $\operatorname{Log}(z)$ is not even continuous along





(i) $\lim_{z \to x_0} \operatorname{Log}(z) = \ln(|x_0|) + i\pi$

(ii) $\lim_{z \to x_0} \operatorname{Log}(z) = \ln(|x_0|) + i(-\pi)$

different

Redefine: $\operatorname{Log}(z) = \ln(|z|) + i\operatorname{Arg}(z), \quad -\pi < \operatorname{Arg}(z) < \pi$

6

Panel 7

Proof of Thm: Easy part: $(2) \Leftrightarrow (3)$

Say: $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ for γ_1, γ_2 from z_1 to z_2:

$\Rightarrow \oint_{\gamma} f(z)\,dz = 0$ take any closed loop

$$\int_{\gamma_L} f(z)\,dz = \int_{-\gamma_R} f(z)\,dz$$

$$\Rightarrow \int_{\gamma_L} f(z) - \int_{-\gamma_R} f(z)\,dz = 0 = \oint_{\gamma} f(z)\,dz$$

7

Panel 8

Now prove that $(1) \Rightarrow (2)$: Say f has antiderivative F.

Path γ from z_1 to z_2, say $z(t)$.

$$\frac{d}{dt} F(z(t)) = F'(z(t)) \cdot z'(t) = f(z(t))\, z'(t)$$

$$\int_{\gamma} f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt = \int_a^b \frac{d}{dt} F(z(t))\,dt = F(z(b)) - F(z(a)) =$$

$$F(z_2) - F(z_1)$$

8

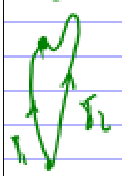
Panel 9

Finally hard part: (3) $\Rightarrow$ (1) : Say $\oint_\gamma f(z)\,dz = 0 \ \forall \gamma$

Want to prove: f has antiderivative.

$$F(z) = \int_{z_1}^{z} f(s)\,ds$$

① It is well-defined: if γ_1, γ_2 both go from z_1 to z then



$$\oint_{\gamma_1 - \gamma_2} f(s)\,ds = 0 \quad \text{by assumption}$$

$$\Rightarrow \int_{\gamma_1} f(s)\,ds = \int_{\gamma_2} f(s)\,ds$$

② Need $F'(z) = f(z)$, Take $F(z+\Delta z) - F(z) = \displaystyle\int_{z}^{z+\Delta z} f(s)\,ds$

Panel 10

Note: $\displaystyle\int_{z}^{z+\Delta z} 1\,ds = \int_0^1 \Delta z\,dt = \Delta z$ $\qquad s(t) = z + t(\Delta z), \ t \in [0,1]$

$ds = \Delta z\,dt$

$$\Rightarrow f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z)\,ds$$

$$\Rightarrow \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(s)\,ds - \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\,ds =$$

$$= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(s) - f(z)\,ds$$

f is continuous $\Rightarrow$ given any $\varepsilon > 0$ $\exists \delta > 0$ s.t. if $|z - s| < \delta \Rightarrow$

$$|f(z) - f(s)| < \varepsilon$$

Panel 11

$$\Rightarrow \underbrace{\frac{F(z+\Delta z)-F(z)}{\Delta z}} - \underbrace{f(z)} = \frac{1}{\Delta z}\int_z f(s)\,ds - \frac{1}{\Delta z}\int_z f(z)\,ds =$$

$$= \frac{1}{\Delta z}\int_z^{z+\Delta z} f(s)-f(z)\,ds$$

f is continuous $\Rightarrow$ given any $\varepsilon > 0$ $\exists \delta > 0$ s.t. if $|z-s|<\delta \Rightarrow$

$$|f(z)-f(s)|<\varepsilon$$

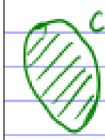
$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| \leq \frac{1}{|\Delta z|}\int_z^{z+\Delta z} |f(s)-f(z)|\,ds \leq \frac{1}{\Delta z}\int_z^{z+\Delta z} \varepsilon\, ds = \varepsilon$$

Thus: $\lim_{\Delta z \to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$ i.e. $F'(z) = f(z)$

qed

Panel 12

Theorem: C is a closed, simple (no intersections) curve and f is analytic inside and on C. Also, f' is continuous there. Then:



C

$$\oint_C f(z)\,dz = 0$$

Recall Green's Theorem: If $P(x,y)$ and $Q(x,y)$ are continuous in D, and all partials exist and are continuous then:

$$\int \vec{F}\cdot d\vec{r} = \int (P,Q)\cdot(dx,dy) = \oint_C P\,dx + Q\,dy = \iint_D Q_x - P_y\,dA$$

$$\int P\,dx = \int P(x(t),y(t))\,x'(t)\,dt$$

$$\int Q\,dy = \int Q(x(t),y(t))\,y'(t)\,dt$$

Panel 13

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt =$$

$$= \int_a^b \left[u(x,y) + i v(x,y)\right]\left[x' + i y'\right] dt =$$

$$= \int_a^b u x' - v y' \, dt + i \int_a^b u y' + v x' \, dt$$

$$= \int_a^b \overset{P}{u}\,dx \overset{Q}{- v}\,dy + i \int_a^b \overset{Q}{u}\,dy + \overset{P}{v}\,dx$$

$$= \iint_D -v_x - u_y \, dA + i \iint_D u_x - v_y \, dA = \underline{0}.$$

$u_y = -v_x$ $\qquad$ $u_x = v_y$

qed

13

Panel 14

<u>Cauchy's Thm</u> f analytic + f' ~~continuous~~ inside C

Then: $\oint_C f(z)\,dz = 0$

<u>Cauchy-Goursat Thm:</u> f analytic on and inside closed curve C

$\Rightarrow \int_C f(z)\,dz = 0$

14

Panel 15

Augustin-Louis Cauchy

Augustin-Louis Cauchy around 1840./ Lithography of Zéphirin Belliard after a painting by Jean Roller.

Born	21 August 1789 Paris, France
Died	23 May 1857 (aged 67) Sceaux, France
Residence	France
Nationality	French
Fields	Infinitesimal calculus Complex analysis

Edouard Goursat

Edouard Goursat

Born	21 May 1858 Lanzac, Lot
Died	25 November 1936
Nationality	France
Fields	mathematics
Alma mater	École Normale Supérieure
Doctoral advisor	Gaston Darboux

15

Panel 16

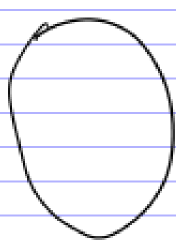
Cauchy - Goursat Theorem (Improved)

If f is analytic in a simply connected domain D. Then

$$\int_C f(z)\,dz = 0 \quad \forall \text{ ~~simple~~ closed curves } C \text{ in } D$$

Corollary: If f is analytic in a simply connected domain D then f has antiderivative.

D



f analytic $\Rightarrow \oint_C f(z)\,dz = 0 \Rightarrow$ f has antider!

16

Panel 17

Cauchy - Goursat Theorem

If f is analytic in a simply connected domain D then

$$\int_C f(z)\,dz = 0 \quad \text{for every closed curve } C \text{ in } D$$

Corollary: If C_1 and C_2 are two simple closed curves, positively oriented, with C_1 inside C_2. Then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

17

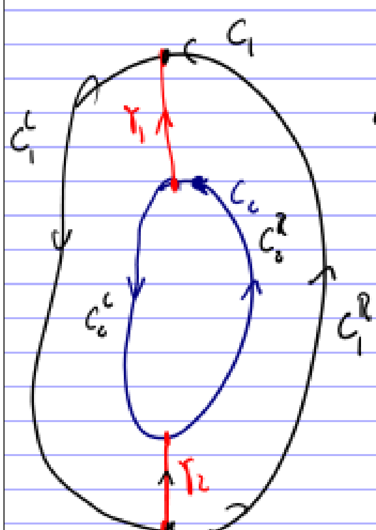
Panel 18

c

Corollary: If C_1 and C_2 are two simple closed curves, positively oriented, with C_1 inside C_2. Then

Deformation Thm

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$



Know $\displaystyle\int_{\gamma_1+C_1^L+\gamma_2-C_2^L} f(z)dz = 0$

$$\int_{C_1^R-\gamma_1-C_2^R-\gamma_2} f(z)dz = 0$$

$$\Bigg\} \int_{C_1} = \int_{C_2}$$

18

Panel 19

Ex: Find $\int_C e^{z^3}\,dz = 0$ where C is square of length 4.

Ex: Find $\int_C \frac{1}{3z}\,dz$ where C square of side 4.

Ex: Find $\int_C \bar{z}\,dz$ where C square of side 4.

19