Panel 1

We defined special functions:

$$e^z = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

unbounded + periodic!

$$\log(z) = \ln|r| + i(\theta + 2k\pi) \qquad z = re^{i\theta}$$

$$\mathrm{Log}(z) = \ln|r| + i\theta$$

1

Panel 2

$u_x = v_y \quad u_y = -v_x \qquad U_x = V_y \quad U_y = -V_x$

$w = v - V$

$w_x = v_x - V_x = -u_y - (-u_y) = 0$

$w_y = v_y - V_y = u_x - u_x = 0$

$w(x, y) = c$

$f = u + iv, \quad F = u + iV$

$f - F = i(v - V) = c$, analytic and purely imag.

2

Panel 3

For extra fun:

① Solve $\cos(z) = 2$, approximate

try: $\cos(it) = \frac{1}{2}\left(e^{i(it)} + e^{-i(it)}\right) = \frac{1}{2}\left(e^{-t} + e^{t}\right) = \cosh(t) = 2$

$t = 1.316957897$

② Find i^{i^i}

$(i^i)^i = i^{-1} = -i$

$i^{(i^i)} = i^{e^{-\pi/2}}$

fsolve(cosh(t) = 2)

1.316957897

cos(I·1.316957897)

2.000000000

$(i^i) = e^{-\pi/2}$

3

Panel 4

$$2^i = e^{i\log(2)} = e^{i(\ln|2| + i(0 + 2k\pi))} =$$

$$= e^{i\ln(2)} e^{-2k\pi}$$

$$= \left[\cos(\ln(2)) + i\sin(\ln(2))\right] e^{-2k\pi}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\log(-2) = \log\left(2e^{i\pi}\right) = \ln|2| + i(\pi + 2k\pi)$$

4

Panel 5

De-Tour Sad to $\mathbb{R}$ ← you thought I forgot ☺

$f:\mathbb{R}\to\mathbb{R}$ s.t. f diff'ble except at $t=0$

$f(x)=x^{1/3}$, $f'=\frac{1}{3}x^{-2/3}$, $x\neq 0$

$f:\mathbb{R}\to\mathbb{R}$ s.t. f diff'ble $\forall t$ but f' is not.

$x^{4/3} \to x^{1/3} \to 1/?$

$f:\mathbb{R}\to\mathbb{R}$ s.t. f diff'ble $\forall t$, f' diff'ble $\forall t$, but f'' not.

$x^{7/3}$

f is diff'ble, f' not cont!

Panel 6

Chapter 4: Integrals.

Integration in Complex Analysis is very important and leads to beautiful, profound theorems with elegant proofs.

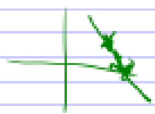
> Pure mathematics is, in its way, the poetry of logical ideas. -Albert Einstein

Before we integrate...

Panel 7

Complex Parametric Functions, $(\cos(t), \sin(t))$

Def: A parametric function $r(t)=(x(t),y(t))$ is a function from $\mathbb{R}$ to $\mathbb{R}^2$. A complex parametric function $w(t)=x(t)+iy(t)$ is a function from $\mathbb{R}\to\mathbb{C}$



Ex: $z(t)=(3+i)+(1-i)t$ line

$z(t)=\cos(t)+i\sin(t)=e^{it}$, circle radius 1, center 0

Panel 8

Derivatives of Complex parametric functions.

Def: If $z(t)=x(t)+iy(t)$ then $\frac{dz}{dt}=x'(t)+iy'(t)$

Ex: If $z(t)=5\cos(3t)+5i\sin(3t)$, find $z'(t)$

$z'(t)=-15\sin(3t)+15i\cos(3t)=$

$z(t)=5e^{i3t}$

$z'(t)=15i\,e^{i3t}$

Panel 9

Integrals of Complex Parameter Functions

Def: If $z(t) = x(t) + i\,y(t)$ then

$$\int z(t)\,dt = \int x(t)\,dt + i\int y(t)\,dt$$

Ex:
$$\int_0^1 (1+it)^2\,dt = \int_0^1 1 - t^2 + 2it\,dt = t - \frac{1}{3}t^3\Big|_0^1 + i\,t^2\Big|_0^1 = \dots$$
$$= 1 - \frac{1}{3} + i$$

Panel 10

Ex:
$$\int_0^\pi e^{it}\,dt = \frac{1}{i}e^{it}\Big|_0^\pi = \frac{1}{i}\left(e^{i\pi} - e^{i0}\right) = -i\left((-1)-1\right) = 2i$$

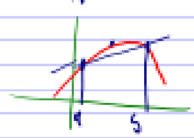
$$\int_0^\pi \cos(t) + i\sin(t)\,dt = \int_0^\pi \cos(t)\,dt + i\int_0^\pi \sin(t)\,dt$$
$$= \sin(t)\Big|_0^\pi - i\cos(t)\Big|_0^\pi =$$
$$= 0 \quad - i(-1-1) = 2i$$

Panel 11

Most rules from diff. and int. apply but not all.

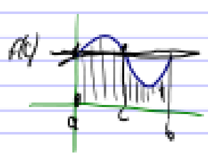
Neither the Mean Value Thm. for Differentiation

$$f'(t) = \frac{f(b)-f(a)}{b-a}$$



nor the Mean Value Thm for Integration

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

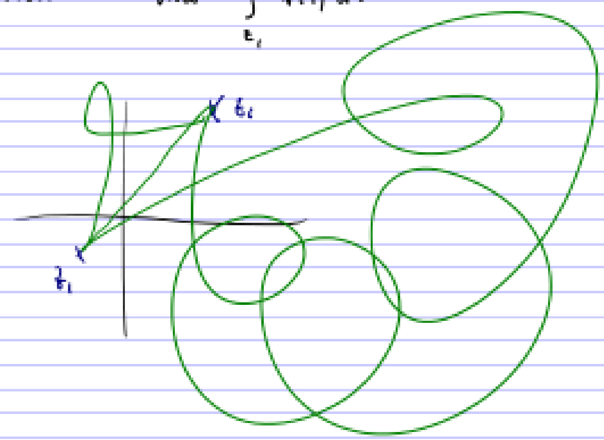


apply for $z(t) = x(t) + iy(t)$

Panel 12

How to define $\int_{z_1}^{z_2} f(z)\,dz$

Problem: How to go from z_1 to z_2?



Panel 13

What is the definition of:

- A complex number
- Adding and Multiplying, Sub and Div, graphically
- Complex roots
- Mapping properties of complex functions
- Arg(z) and arg(z)
- The limit of a complex function f(z) as z approaches c is L
- Continuity of a complex function f(z) at a point z = c
- The complex derivative of a function f(z)
- Analytic function
- CR equations
- Entire function
- Harmonic conjugate of a function u
- Harmonic function
- e^z, sin(z), cos(z), log(z), and Log(z)
- Euler's Formula
- Functions z(t), its integral and derivative
- Different paths (line segments and circles)
- Contour Integrals

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