Panel 1

Complex Concepts

$f: D \subset \mathbb{C} \to \mathbb{C}$

Limits: $\lim_{z \to z_0} f(z) = L$ if $z \in D_\delta(z_0) \Rightarrow f(z) \in D_\varepsilon(L)$

Continuity: $\lim_{z \to z_0} f(z) = f(z_0)$

$\mathbb{C}$-Diff'ble: $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$

Analytic: f is analytic at z_0 if f is $\mathbb{C}$-diff'ble in some $D_r(z_0)$

Entire: analytic in $\mathbb{C}$

Harmonic: $f: \mathbb{R}^2 \to \mathbb{R}^2$ is harmonic if $\Delta f = f_{xx} + f_{yy} = 0$

Laplacian operator

1

Panel 2

CR: f is $\mathbb{C}$-diff'ble $\Leftrightarrow$

$$u_x = v_y$$
$$u_y = -v_x$$
$$f'(z) = u_x + i v_x$$

Theorems:

$$\left.\begin{array}{l} f \text{ analytic } \& \ f'(z) \equiv 0 \\ f \text{ analytic}, \ |f(z)| \equiv c \\ f \ \& \ \bar{f} \text{ both analytic} \end{array}\right\} f \text{ is const.}$$

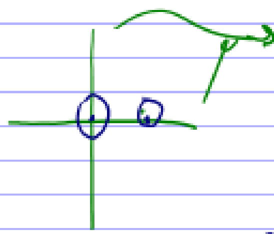
2

Panel 3

Ex: Is $f(z) = \underbrace{x^3 + 3xy^2}_{u} + i\,(\underbrace{y^3 + 3x^2y}_{v})$ diff'ble anywhere?
Is it analytic anywhere?

① $u_x = 3x^2 + 3y^2 \stackrel{!}{=}$ ② $v_y = 3y^2 + 3x^2$

$u_y = 6xy = -\, v_x = 6xy \Rightarrow x = 0,\ y = 0$



f is $\mathbb{C}$-diff'ble on $\{x=0\}$, $\{y=0\}$

f is not analytic anywhere

Ex: $f(z) = z^3 + 3z^2 - 4z + 1$ is entire

3

Panel 4

Ex: Which of these are harmonic functions:

$u(x,y) = x^3 - 3xy^2$

$u_x = 3x^2 - 3y^2$ $u_{xx} = 6x$

$u_y = -6xy$ $u_{yy} = -6x$ ⊕ $\Delta u = 0$ ✓

$v(x,y) = 3x^2y + y^3$

$u_x = 6xy$ $u_{xx} = 6y$

$u_y = 3x^2 + 3y^2$ $u_{yy} = 6y$ ⊕ $\Delta u \neq 0$ harmonic if $y=0$

$T(x,y) = e^{-y}\sin(x)$

HW

4

Panel 5

Theorem: If f is analytic in a domain D and $f(z) = u(x,y) + iv(x,y)$ then u, v are both harmonic.

(Proof)

$$u_x = v_y \Rightarrow u_{xx} = v_{yx}$$
$$u_y = -v_x \Rightarrow u_{yy} = -v_{xy}$$
$$u_{xx} + u_{yy} = 0 \Rightarrow u \text{ is harmonic}$$

Harmonic functions show up in Physics a lot.

5

Panel 6

Def: If $u(x,y)$ is harmonic and $v(x,y)$ is also harmonic s.t. $u+iv$ is analytic, then u and v are called harmonic conjugates.

$$f(x,y) = \underbrace{x^2-y^2}_{u} + \underbrace{2ixy}_{v}$$ u, v are harmonic conjugates.

6

Panel 7

[simply connected]

Thm: If $u(x,y)$ is harmonic in a 'special' domain D then u has a harmonic conjugate

Ex: $u(x,y) = x^3 - 3xy^2$ is harmonic. Find harmonic conjugate!

$u_x = 3x^2 - 3y^2 = v_y$

$v = \int 3x^2 - 3y^2 \, dy = 3x^2 y - y^3 + C(x)$

$v_x = 6xy + C'(x) \overset{!}{=} -u_y = -(-6xy) \Rightarrow C'(x) = 0, \ C = \text{constant}$

$v = 3x^2 y - y^3 + C \quad \Rightarrow f(z) = x^3 - 3xy^2 + i(3x^2 y - y^3) + C$

$= z^3$

7

Panel 8

Ex: $u(x,y) = e^{-y}\sin(x)$. Find $v(x,y)$ such that $f(z) = u(x,y) + i v(x,y)$ is analytic. Find $f'(z)$

$u_x = e^{-y}\cos(x) = v_y$

$v = -e^{-y}\cos(x) + C(x) \Rightarrow v_x = e^{-y}\sin(x) + C'(x) = -u_y = -e^{-y}\sin(x) \Rightarrow C = \text{const}$

$\Rightarrow f(z) = e^{-y}\sin(x) - ie^{-y}\cos(x) \overset{!}{=} e^{iz} = -ie^{iz}$

$f'(z) = e^{-y}\cos(x) + ie^{-y}\sin(x) = e^{-y}(\cos(x) + i\sin(x)) = e^{-y}e^{ix} = e^{-y+ix} = e^{i(x+iy)}$

$= e^{iz}$

$f(z) = iy \quad , \ f(z) = iz^2$

8

Panel 9

Chapter 3: exp and Friends

Def: $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$

Properties:

(i) $|e^z| = |e^x e^{iy}| = e^x |e^{iy}| = e^x \cdot 1 = e^x$

(ii) $e^z \neq 0$ $\quad e^z = 0 \iff e^x(\cos(y) + i\sin(y)) = 0$, $e^x \neq 0$; $\cos(y) = 0 \Rightarrow y = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$; $\sin(y) = 0$, $y = k\pi$ — never simult.

(iii) e^z is periodic $2\pi i$ because $e^{z+2\pi k i} = e^{x+i(y+2k\pi)} = e^x e^{iy} \underbrace{e^{i2k\pi}}_{1} = e^z$

(iv) $\frac{d}{dz} e^z = e^z$

9

Panel 10

Ex: Solve $e^z = 1+i$

$e^x(\cos(y) + i\sin(y)) = 1+i$ $\quad e^x\cos(y) = 1$, $e^x\sin(y) = 1$

$e^z = 1+i = \sqrt{2}e^{i\pi/4}$

$e^x e^{iy} = \sqrt{2}e^{i\pi/4}$ $\quad \Rightarrow e^x = \sqrt{2}$, $y = \frac{\pi}{4} + 2k\pi$

$x = \ln(\sqrt{2})$

10

Panel 11

Def: If $z = re^{i\theta}$, define $\log(z) = \ln(r) + i(\theta + 2k\pi)$

$$e^{\log(z)} = e^{\ln(r) + i(\theta + 2k\pi)} = e^{\ln(r)} e^{i\theta} e^{i2k\pi}$$
$$= re^{i\theta} = z$$

Note: $\log(z)$ is not a function (multivalued)

Note: $\log(e^z) = \log(e^x e^{iy}) =$
$$= \ln(e^x) + i(y + 2k\pi) = x + i(y + 2k\pi) \neq z \quad (\text{only } k = 0!)$$

11

Panel 12

Ex: $\log(1) = \log(1e^{i0}) = \ln(1) + i(0 + 2k\pi) = i2k\pi$

Def: The principle value of the Logarithm is:
$$\text{Log}(z) = \ln(r) + i\theta \quad , \text{if } z = re^{i\theta} \quad (\to \text{Log}(1) = 0)$$

Properties:

$$\frac{d}{dz} \log(z) = \frac{1}{z}$$

12

Panel 13

Ex: $\ln(x)$ is really defined for all $x \neq 0$

$$\log(-1) = \log(1e^{i\pi}) = \ln(1) + i(\pi + 2k\pi) = i(\pi + 2k\pi)$$

$k=0$

$$\mathrm{Log}(-1) = \underline{i\pi}$$

13

Panel 14

<u>Trig Functions:</u>

$$e^{it} = \cos(t) + i\sin(t)$$

$$e^{-it} = \cos(-t) + i\sin(-t) = \cos(t) - i\sin(t)$$

$$\frac{1}{2}\left(e^{it} + e^{-it}\right) = \cos(t) \qquad t \in \mathbb{R}$$

$$\frac{1}{2i}\left(e^{it} - e^{-it}\right) = \sin(t)$$

14

Panel 15

Complex Trig Functions:

Def: $$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

Ex: $$\sin(i) = \frac{1}{2i}\left(e^{i^2} - e^{-i^2}\right) = \frac{1}{2i}\left(e^{-1} - e^{1}\right)$$

$$\lim_{t \to \infty} \cos(it) = \lim_{t \to \infty} \frac{1}{2}\left(e^{i(it)} + e^{-i(it)}\right) = \lim_{t \to \infty} \frac{1}{2}\left(e^{-t} + e^{t}\right) = \infty$$

15