Panel 1

Last Time

Analytic vs diff'ble functions

diff'ble $\forall z$ in $D_r(z_0)$

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) \text{ at } z_0$$

Various Theorems

f analytic at z_0

$f, \bar{f}$ analytic $\Rightarrow f$ const

f analytic, $\|f\| = \text{const} \Rightarrow f$ const

f analytic, $f \in \mathbb{R} \Rightarrow f$ const

$f' = 0 \ \forall z \in D \Rightarrow f$ const

Entire: f analytic $\forall z \in \mathbb{C}$

Ex: $\cancel{\frac{1}{1-z}}$

$\cancel{\frac{1}{1+z^2}}$

e^z

1

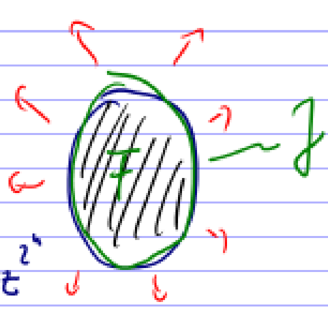
Panel 2

Iterations of functions $f_c(z) = z^2 + c$

seed z_0, $O_f(z_0) = \{z_0, f(z_0), f(f(z_0)), f(f(f(z_0))), \ldots, f^{(n)}(z_0), \ldots\}$

Fatou set: $F_{f_c} = \{z : \|f^{(n)}(z)\| \text{ is bounded}\}$

Julia set: $J_{F_c} = \partial F_{F_c}$



$c = 0$:
$f(z) = z^2$
$f^{(n)}(z) = z^{2^n}$

Douady's Rabbit

Seahorse, ...

2

Panel 3

Theorem: If $f_c(z) = z^2 + c$, then F_{f_c} is not empty!

$f_c(z) = z \iff z^2 + c = z$ has 2 solutions

fixed points. $\in F_{f_c}$

$z \rightsquigarrow f(z) \rightsquigarrow f(f(z))$

$z \rightsquigarrow$

$f_c(f_c(z)) = z \iff (z^2+c)^2 + c = z$ has 4 solution

periodic points of period 2 $\in F_{f_c}$

$f^{(n)}(z) = z$ $\quad z^{2^n}$ degree $\Rightarrow$ has 2^n solutions

periodic n-points $\in F_{f_c}$

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Panel 4

$f = u + iv$ analytic $\Rightarrow$ u, v are harmonic, $u_{xx} + u_{yy} = 0$

f is 2-cont. diff'ble, same u, v.

$$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right| \begin{array}{l} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{array} \quad \left.\begin{array}{l} u_{xx} = v_{yx} \\ u_{yy} = -v_{xy} \end{array}\right\} \text{ are equal if } v \text{ is 2-times cont. diff'ble}$$

$$u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0 \Rightarrow u \text{ harmonic.}$$

v is similar

$-\frac{1}{2} + i \rightsquigarrow -\frac{1}{2} - i \rightsquigarrow -\frac{1}{2} + i \rightsquigarrow$

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Panel 5

<u>Note:</u> All periodic points are in the Fatou set.

<u>Thm:</u> Say z_0 is a fixed point, i.e. $f_c(z_0) = z_0$:

if $|f_c'(z_0)| < 1 \Rightarrow$ attracting fixed point

if $|f_c'(z_0)| > 1 \Rightarrow$ repelling fixed point.

<u>Ex:</u> If $f_c(z) = z^2 - 1/2$ the fixed points are?

fixed: $f(z) = z \Rightarrow z^2 - \frac{1}{2} = z \Rightarrow z^2 - z - \frac{1}{2} = 0$, $z_{1/2} = \frac{1 \pm \sqrt{1+2}}{2} = \frac{1 \pm \sqrt{3}}{2}$

$f'(z) = 2z = 1 \pm \sqrt{3} \Rightarrow 1 + \sqrt{3} > 1 \Rightarrow$ <u>repelling</u>

$1 - \sqrt{3} < 1 \Rightarrow$ attracting

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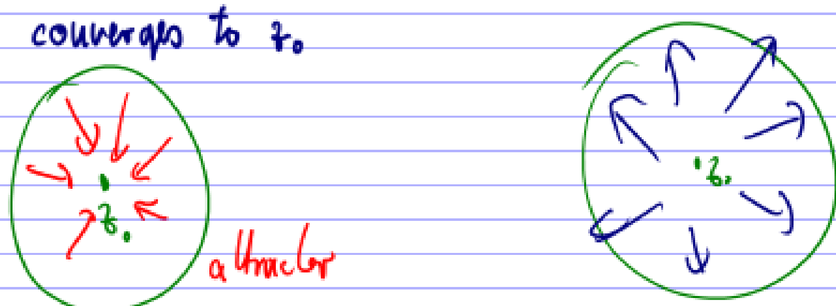
Panel 6

<u>Note:</u> If z_0 is an attracting fixed point, then there is a nbhd $D_r(z_0)$ of z_0 s.t. orbit of all $z \in D_r(z_0)$ converges to z_0



<u>Ex:</u> If $f_c(z) = z^2 - 1$, are the fixed points attracting or repelling? How about period 2 points?

$z^2 - 1 = z \Rightarrow z_{1/2} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$, $f'(z) = 2z = 1 \pm \sqrt{5}$ repelling

$f(f(z)) = (z^2-1)^2 - 1 = z$ $\quad z^4 - 2z^2 - z = 0$, $z(z^3 - 2z - 1) = 0$ $\quad z = 0$, $z = -1$

$z^4 - 2z^2 + \cancel{1} - \cancel{1} = z$ $\quad \frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$

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Panel 7

Thm: $\mathcal{F}_{f_c}$ is either connected or totally disconnected

(has no interior)

Def:

$M = \{ c : \mathcal{F}_{f_c} \text{ is connected} \}$ $\quad f_c(z) = z^2 + c$

Thm: $M \neq \emptyset$ $\quad c = 0 \in M$, because $\mathcal{F}_{f_0}$ is unit disk

$c = -1 \in M$ because attracting period-2 point

$\Rightarrow \mathcal{F}$ can't be totally disconnected

$c = \frac{1}{4} \in M$

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Panel 8

$M = \{ c : \mathcal{F}_{f_c} \text{ is connected} \}$, $f_c(z) = z^2 + c$

is called Mandelbrot set (Benoit Mandelbrot)

Thm: If orbit of $z = 0$ is bounded for $f_c(z) = z^2 + c$

then $\mathcal{F}_{f_c}$ is connected and $c \in M$

$0 \in M$: $\mathcal{O}_{f_0}(0) = \{0, 0, 0, \ldots\}$ bdd. $\quad f_0(z) = z^2$

$1+i \in M$? $\mathcal{O}_{f_c}(0) = \{0, (1+i), (1+i)^2 + (1+i), \ldots\}$ $\quad f_c(z) = z^2 + (1+i)$

bdd or not?

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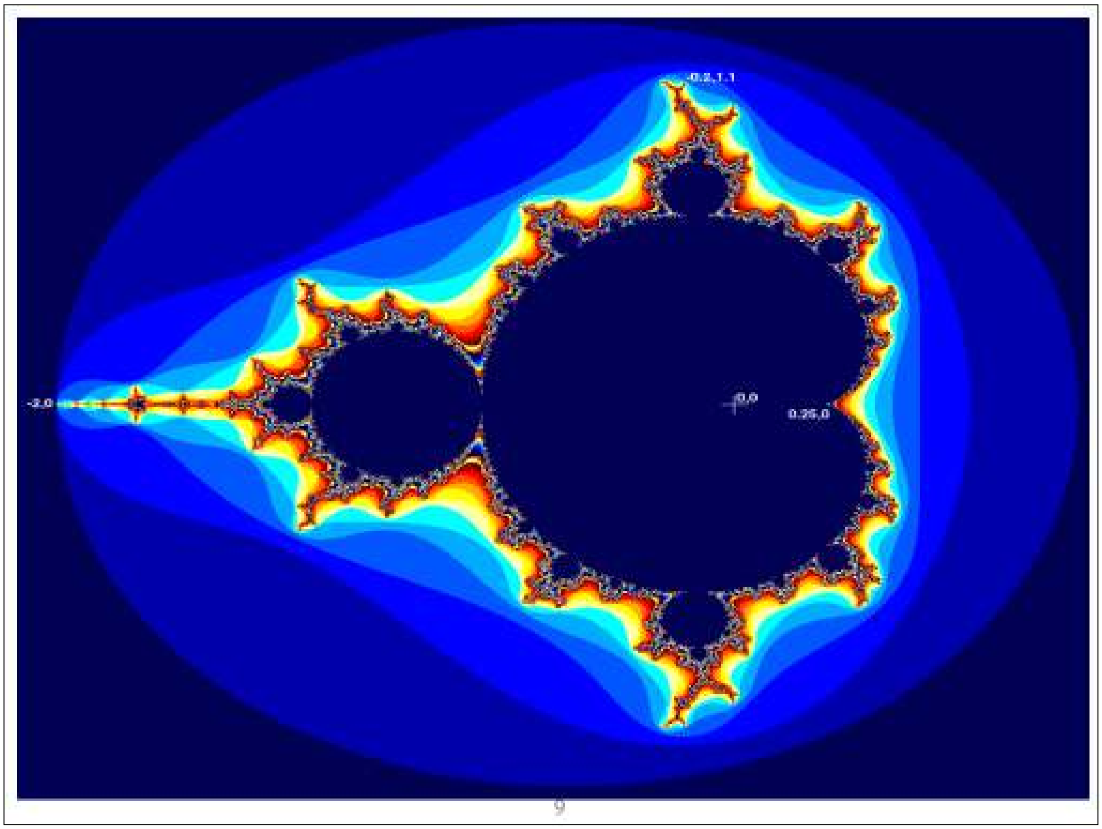
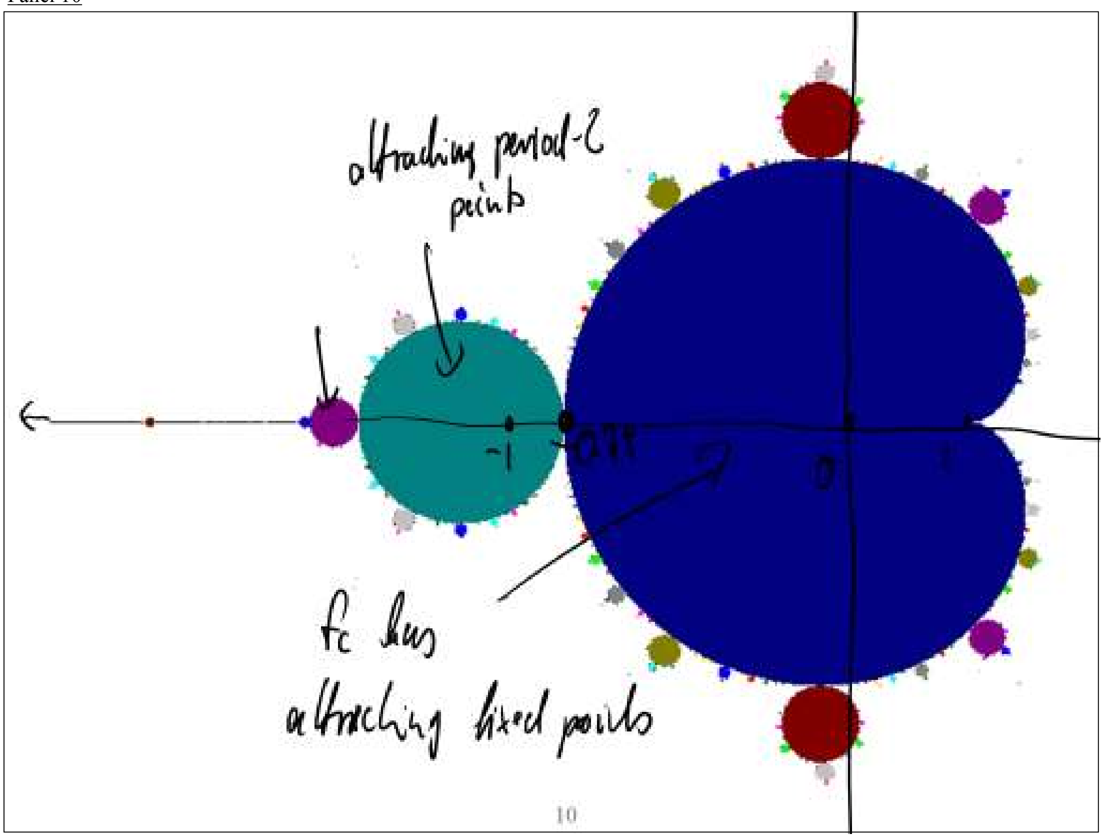
Panel 9



Panel 10



Panel 11

For which $c \in \mathbb{R}$ does $f_c(z) = z^2 + c$ have an attractive fixed point?

$$f(z) = z \Leftrightarrow z^2 + c = z \Rightarrow z^2 - z + c = 0 \qquad z_{1,2} = \frac{1 \pm \sqrt{1-4c}}{2}$$

$$|f'(z)| = |2z| = |1 \pm \sqrt{1-4c}| < 1$$

$c \in \mathbb{R}$ $\qquad 1 \pm \sqrt{1-4c} = 1 \qquad c = \frac{1}{4}$

$$1 \pm \sqrt{1-4c} = 1 - 2$$

$$1 - 4c = 2 \qquad c = -\frac{3}{4} = \underline{\underline{-0.75}}$$

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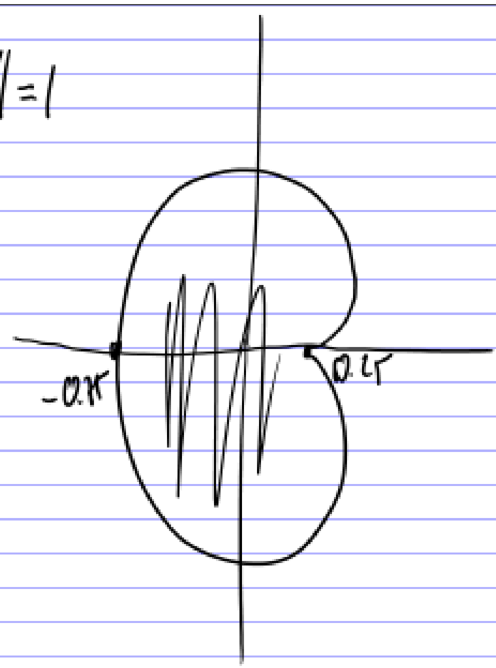
Panel 12

$$\|1 \pm \sqrt{1-4c}\| = 1$$



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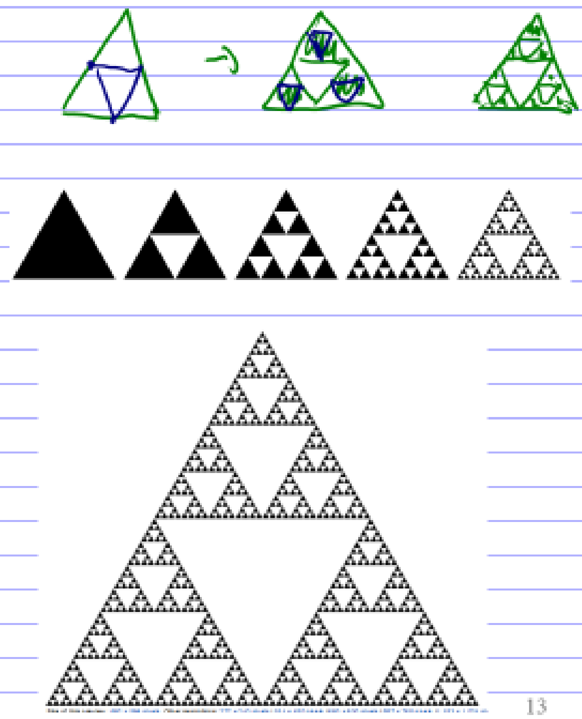
Panel 13

Idea of "Iterating" is Powerful

Sierpinski Triangle



area is zero

bdry is infinitely long!

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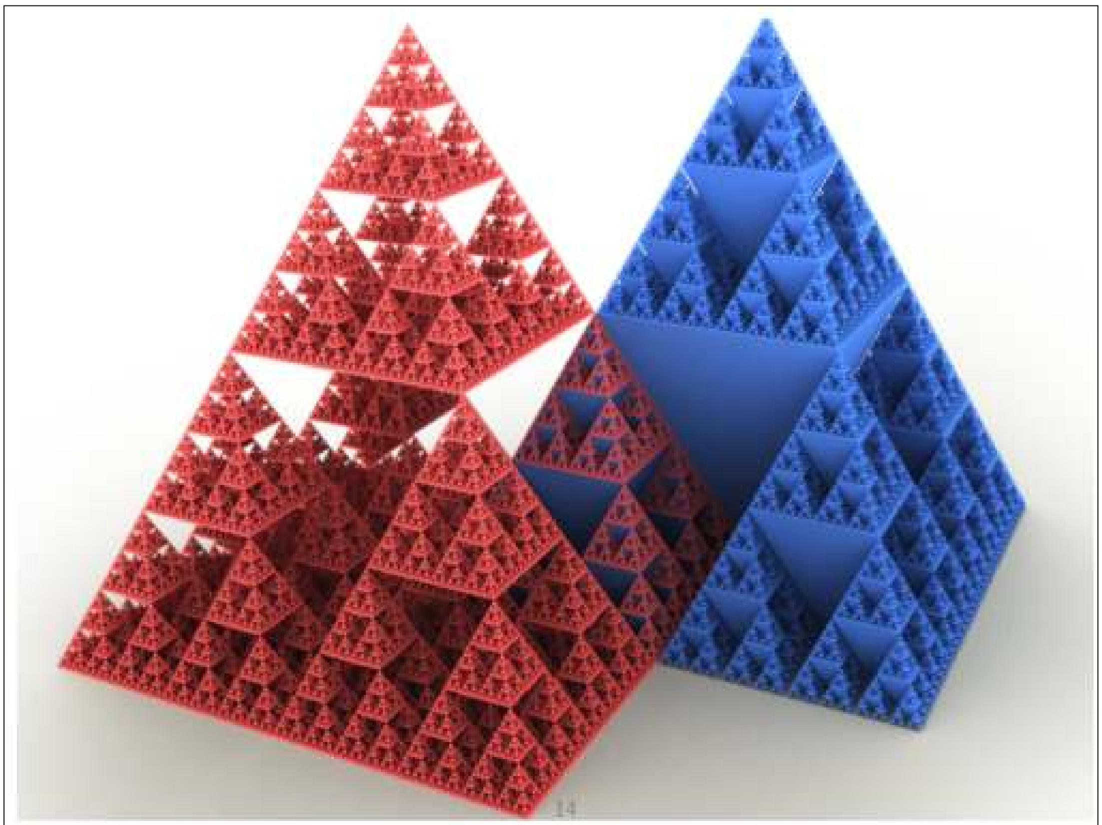
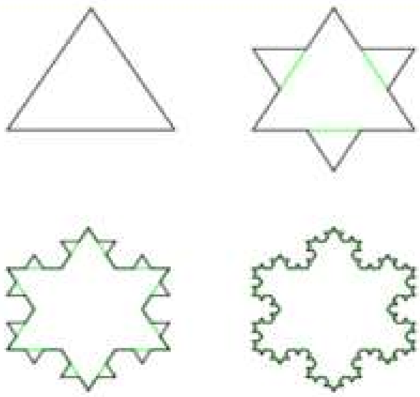
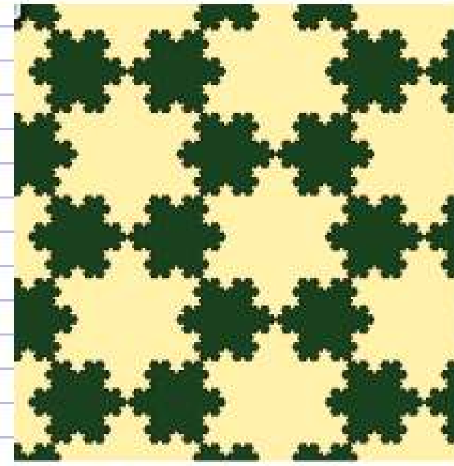
Panel 14



Panel 15





Koch Snowflake

infinite perimeter,
dimension 1.26

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Panel 16

http://upload.wikimedia.org/wikipedia/commo…

Midpoint Displacement Method
resulting in a
Fractal Mountain

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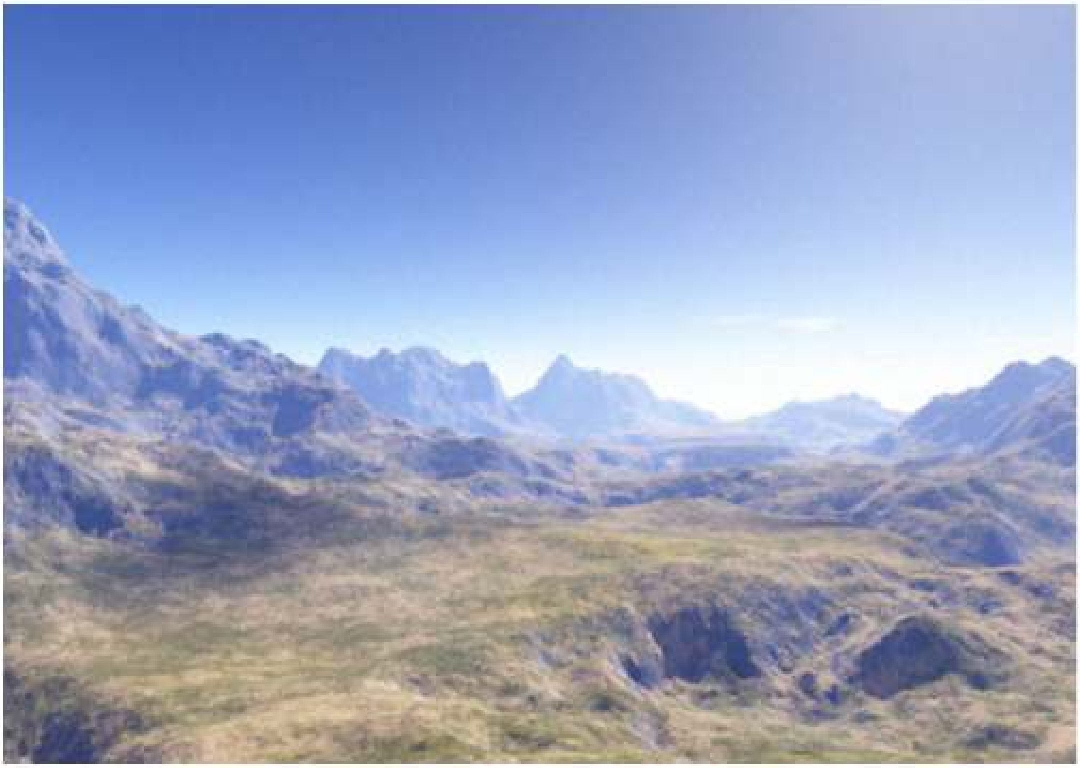
Panel 17



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Panel 18

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Panel 19