Panel 1

Last Time

$\mathbb{C}$ - diff'ble: $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ $\lim_{z \to z_0}$ $|f'(z_0)|$

$\mathbb{C}$ - diff'ble $\Rightarrow$ continuity : $|f(z) - f(z_0)| = \frac{|f(z) - f(z_0)|}{|z - z_0|} |z - z_0| \to 0$

Functions in z are usually diff'ble

regular rules (prod., quotient, chain Rule) apply

CR Equations: $f(z) = u(x,y) + iv(x,y)$

$u_x = v_y$

$u_y = -v_x$

$f'(z) = u_x + iv_x$

$r u_r = v_\theta$

$u_\theta = -r v_r$

rarely used

1

Panel 2

Assume $f(z)$ diff'ble. $f(z) \in \mathbb{R} \ \forall z \in D$

$u_x = v_y$, $-v_x = u_y$ $\operatorname{Im}(f(z)) = 0$, $v_x = v_y = 0$

$v_y = 0$, $u_x = 0$ $v_x = 0$, $u_y = 0$ $u_x = u_y = 0$

u = constant. if u is constant, +

$v = 0$, $f(z) = u$. QED

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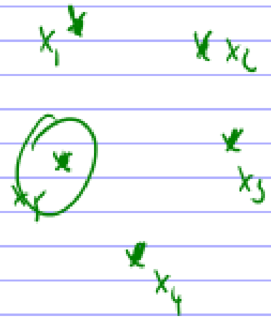
Panel 3



Define $\delta = \min\left(|z_i - z_j|, i \neq j\right)$

$\delta \neq 0$ because set is finite

take any z_j, pick $D_{\frac{\delta}{2}}(z_j) \Rightarrow$ does not contain any z_k other than z_j

$\Rightarrow z_j$ is no accumulation point

$\Rightarrow$ set of accum. points is $\emptyset \subset D \Rightarrow D$ closed

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Panel 4

$\{1, 2, 3, 4, 5, \ldots\}$ is infinite, and closed

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ not closed

$\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ closed

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Panel 5

Theorem: (l'Hospital's Rule)

Prove that if $f(z_0) = g(z_0) = 0$ and $f'(z_0)$, $g'(z_0)$ exist with $g'(z_0) \neq 0$ then $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$

$$\lim_{z \to z_0} \frac{\frac{f(z) - \overset{0}{f(z_0)}}{z - z_0}}{\frac{g(z) - \underset{0}{g(z_0)}}{z - z_0}} = \frac{f'(z_0)}{g'(z_0)}$$

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Panel 6

Analytic Functions

We want to expand on the idea of $\mathbb{C}$-diffble.

Def. A function $f: D \to \mathbb{C}$ is analytic at a point $z_0 \in D$ if f is $\mathbb{C}$ diffble in some neighborhood of z_0

Ex: $f(z) = |z|^2$ was $\mathbb{C}$-diffble only at 0, but not analytic at $z_0 = 0$, or anywhere else

$f(z) = 1/z$ not analytic at $z = 0$, but everywhere else it is.

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Panel 7

Def. If a function $f: \mathbb{C} \to \mathbb{C}$ is analytic in all of $\mathbb{C}$ then f is called entire

Ex: Any polynomial in z is entire

e^z is entire

$\cosh(x) = \frac{e^x + e^{-x}}{2}$

$\sinh(x) = \frac{e^x - e^{-x}}{2}$

$f(z) = \underline{\cosh(x)\cos(y)} + i\,\underline{\sinh(x)\sin(y)}$

$u_x = \sinh(x)\cos(y)$ ✓ $v_y = \sinh(x)\cos(y)$

$u_y = -\cosh(x)\sin(y) = -v_x = \cosh(x)\sin(y)$

CR are true $\forall (x,y)$ $\Rightarrow$ entire

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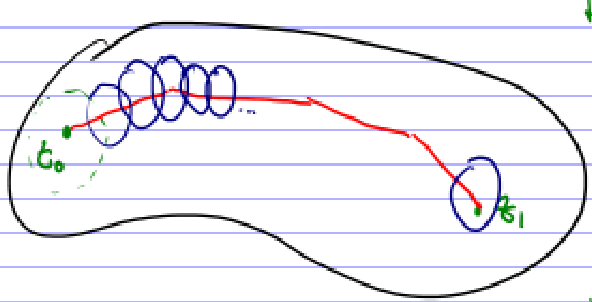
Panel 8

Thm: If f is analytic in a domain D and $f'(z) = 0$ for all $z \in D$, then f is constant.

[Proof] take $z_0 = x + iy$ $\Rightarrow u_x = u_y = v_x = v_y = 0 \quad \forall (x,y) \in D(z_0)$

take any other point $z_1 \in D$



find sequence of lines in D connecting z_0 with z_1

Find finite sequence of disks covering the line. Since disks overlap, f must be the same constant along line, so at z_1.

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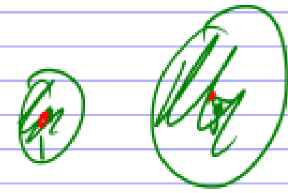
Panel 9

Prev. proof is backward. Want avoid

$$f(z) = \begin{cases} 1 & \text{in } |z|<1 \\ 2 & \text{in } |z-5|<1 \end{cases}$$



$\Rightarrow$ f analytic, $f' \equiv 0$, but f not const.

D is not a domain

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Panel 10

<u>Theorem:</u> If $f(z)$ and $\overline{f(z)}$ are both analytic, then f must be constant.

$f = u + iv$ is analytic $\Rightarrow$

$u_x = v_y$ (1)

$u_y = -v_x$ (2) $\Rightarrow u_x = 0$

$\bar{f} = u + i(-v)$

$u_x = -v_y$ (3) $\quad u_y = 0$

$u_y = v_x$ (4)

$\Rightarrow$ f is constant.

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Panel 11

Theorem: Suppose $f(z)$ is analytic in a domain D and $|f(z)| = \text{const.}$ in D. Then f is constant.

$|f(z)|^2 = c^2 \Rightarrow f\bar{f} = c^2$

$|z|^2 = |z\bar{z}|$

$f\bar{f} = |f(z)|^2 = c^2$

a) If $c = 0 \Rightarrow f(z) = 0$, i.e. const.

b) Assume $c \neq 0 \Rightarrow f, \bar{f}$ are zero for any z

f is analytic

$\bar{f} = c^2/f$ is analytic because $f \neq 0$ and analytic

See last slide $\Rightarrow f = \text{const}$

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Panel 12

Iterative Dynamic Systems: $f(z)$ is some function

Define $z_{n+1} = f(z_n)$, for some starting point z_0

Ex: $f(z) = z^2 - 1$

Iterative dynamic system studies behavior of sequences

$z_0, z_1 = f(z_0), z_2 = f(z_1) = f(f(z_0)), z_3 = f(z_2) = f(f(f(z_0))) \ldots$

Take $z_0 = 0$. $z_1 = f(0) = -1$, $z_2 = f(z_1) = f(-1) = 0$,

$z_3 = f(z_2) = f(0) = -1$, $z_4 = 0$, $z_5 = -1$, $z_6 = 0 \ldots$

$0 \to -1 \to 0 \to -1 \to 0 \to -1 \to \ldots$

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Panel 13

Def: If $f(z)$ is a function, then the sequence of iterates of the starting point z_0 (seed)

$z_0,\ z_1 = f(z_0),\ z_2 = f(z_1), \ldots$

is called the orbit of z_0, $O_f(z_0)$

Ex: Take $f(z) = z^2 - 1$. Find

a) $O_f(0)$: $0 \to -1 \to 0 \to -1 \to 0 \to -1 \ldots$ $\qquad O_f(i): i \to -2 \to 3 \to 8$

b) $O_f(2)$: $2 \to 3 \to 8 \to 63 \to \ldots$ ∞

c) $O_f(1/2)$: $\frac{1}{2} \to -\frac{3}{4} \to -\frac{7}{16} \to \ldots$

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Panel 14

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Panel 15

Orbits are easy to compute with a Computer Program: $f_c(z) = z^2 + c$

Pseudo Code:

Start with z_0
fix function ($c = \#$)
repeat:
compute $f(z)$
assign that to z
print value

```
z := 0.5;
c := -1.0;
for i from 0 to 10 do
  w := z·z + c:
  print(w):
  z := w:
end do:
```

```java
double z = 0.5;
double c = -1.0;

for (int i = 0; i < 10; i++)
{
    z = z*z + c;
    System.out.println(z);
}
```

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Panel 16

Two Basic Questions for Iterative Dynamic Systems

$z_0, z_1 = f_c(z_0), z_2 = f_c(z_1), z_3 = f_c(z_2), \ldots$

where $f_c(z)$ is a function depending on parameter c, such as $f_c(z) = z^2 + c$ or $f_c(z) = c \cdot \sin(z)$

Question A: fix c, what happens to $O_{f_c}(z_0)$ for all z_0

Question B: fix z_0, what happens to $O_{f_c}(z_0)$ for all c

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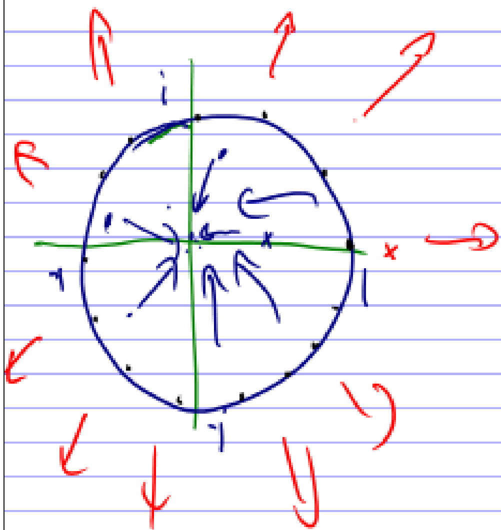
Panel 17

Example. $f_c(z) = z^2 + c$ for $c = 0$. $f(z) = z^2$



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Panel 18

Fatou and Julia Sets

Def: Let $f_c(z)$ be a function with a parameter c.
Define
$$F_c = \{ z_0 : O_{f_c}(z_0) \text{ are bounded} \}$$
Fatou set

Def: Let F_c be the Fatou set for a function f_c.
Then
$$J_c = \partial F_c$$
Julia set

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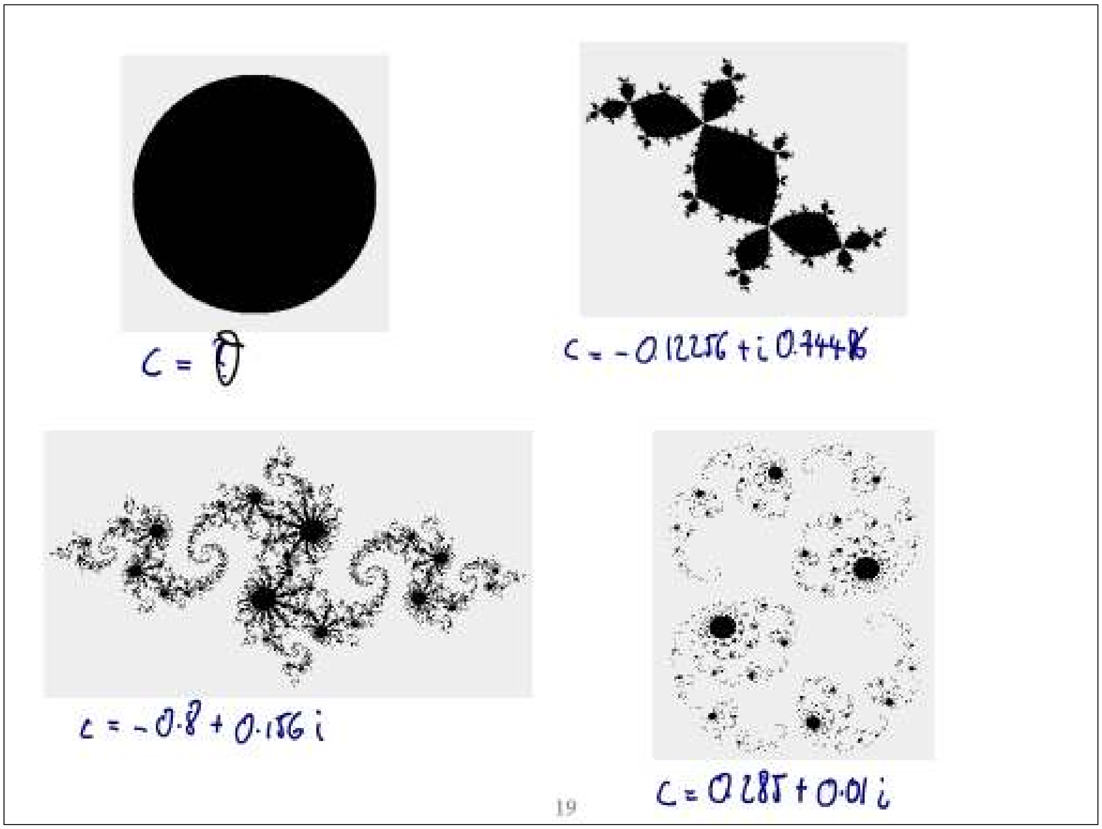
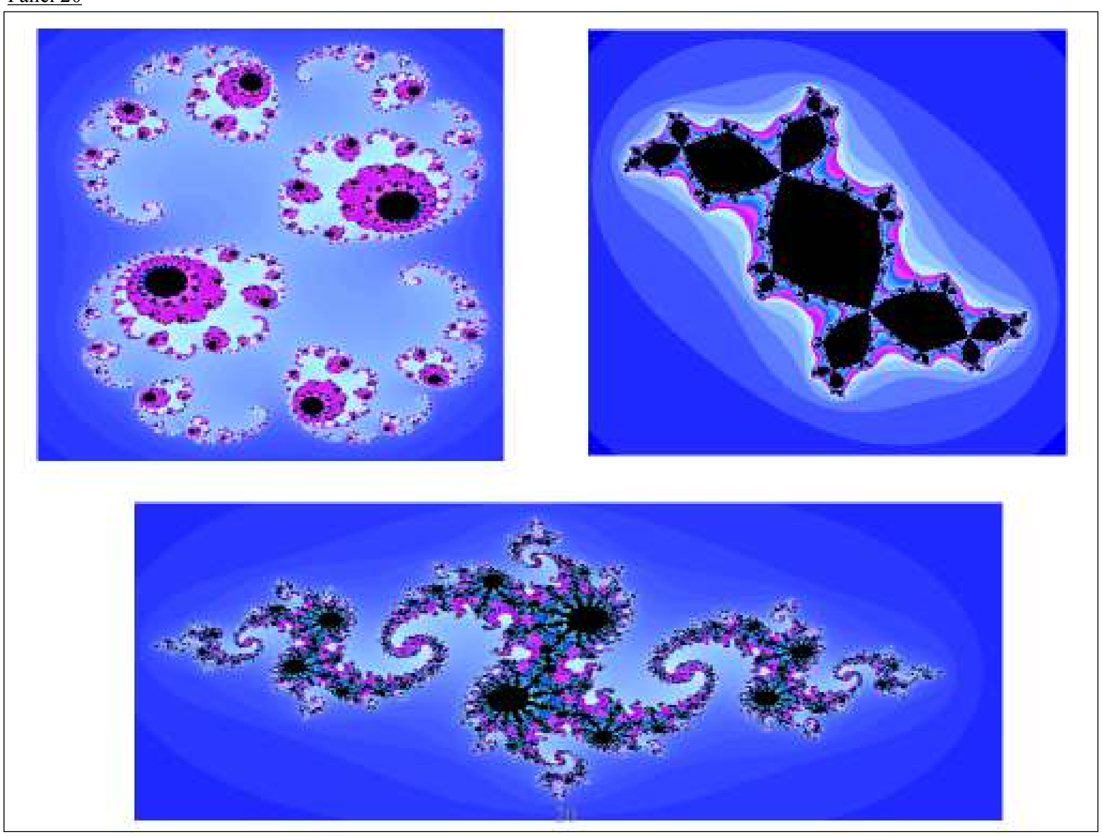
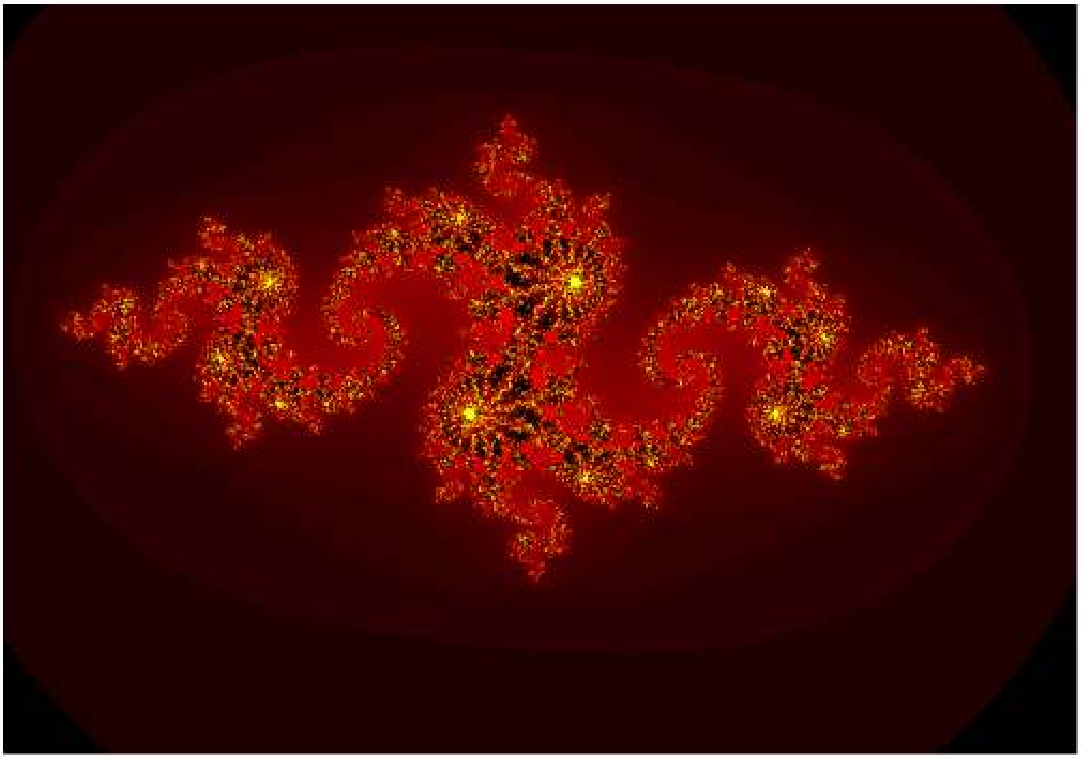
Panel 19



Panel 20



Panel 21



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Panel 22

http://www.youtube.com/watch?v=gruJ0S3TTtI

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