Panel 1

$$\lim_{z\to 0}\left(\frac{z}{\bar z}\right)^2 = \left(\frac{x+yi}{x-yi}\right)^2$$

$$\lim_{\substack{x=0\\ y\to 0}}\left(\frac{\cancel{y}i}{-\cancel{y}i}\right)^2 = (-1)^2 = 1$$

$$\lim_{x=y}\left(\frac{x+xi}{x-xi}\right)^2 = \frac{\cancel{x^2}+2x^2i-\cancel{x^2}}{\cancel{x^2}-2x^2i-\cancel{x^2}} = -1$$

$$-1 \neq 1 \quad \therefore \text{ DONE}$$

1

Panel 2

$$\lim_{z\to i} 3z+1 = 3i+1$$

$$|3z+1-(3i+1)| = |3z-3i| = |3|\cdot|z-i| < \varepsilon$$

$$\delta = \frac{\varepsilon}{3}$$

$\varepsilon > 0$, $\delta > 0$ the limit exists $\delta = \frac{\varepsilon}{3}$

$$|3z+1-3i-1| = 3|z-i| < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

Take $\varepsilon > 0$, let $\delta = \varepsilon/3$. Then if $|z-i| < \delta$

$\Rightarrow$

2

Panel 3

HW

(1) Prove that any finite set in $\mathbb{C}$ is closed.

(2) Find a sequence of inf. points that is also a closed set.

No class on Monday.

3

Panel 4

Derivatives For $f(z)$ a complex function defined in a nbhd of z_0 we define

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If limit exists, f is $\mathbb{C}$- diff'ble

Ex: If $f(z) = z^2$, show that $f'(z) = 2z_0$

$$f'(z_0) = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} \frac{(z-z_0)(z+z_0)}{(z-z_0)} = 2z_0$$

$$\lim_{z \to i} z^2 + 1 = 0$$

$$f(z) = z^3 \Rightarrow f'(z) = 3z^2$$

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Panel 5

Ex: Show that $f(z) = \bar{z}$ is nowhere differentiable!!

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x,y) \to (x_0,y_0)} \frac{\overline{(x+iy)} - \overline{(x_0+iy_0)}}{(x+iy) - (x_0+iy_0)}$$

$$= \lim_{(x,y) \to (x_0,y_0)} \frac{x - x_0 - i(y-y_0)}{x - x_0 + i(y-y_0)}$$

① Let $x = x_0$, $y \to y_0$ $\quad \lim \frac{-i(y-y_0)}{i(y-y_0)} = -1$

② Let $y = y_0$, $x \to x_0$ $\quad \lim \frac{x-x_0}{x-x_0} = +1$

$\Rightarrow f(z) = \bar{z}$ is not diff'ble at any point!

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Panel 6

Ex: Let $f(z) = |z|^2$. Is it differentiable? Where?

$$\lim_{(x,y) \to (x_0,y_0)} \frac{(x^2+y^2) - (x_0^2+y_0^2)}{(x+iy) - (x_0+iy_0)} = \lim_{(x,y) \to (x_0,y_0)} \frac{x^2 - x_0^2 + y^2 - y_0^2}{x - x_0 + i(y-y_0)}.$$

$x = x_0$: $\lim \frac{y^2 - y_0^2}{i(y-y_0)} = \lim \frac{\cancel{(y-y_0)}(y+y_0)}{i\cancel{(y-y_0)}} = -i\,2y_0$ $\quad \left(\frac{1}{i} = -i\right)$

$y = y_0$: $\lim \frac{x^2 - x_0^2}{x - x_0} = 2x_0$

same if $x_0 = y_0 = 0$

$|z|^2$ is $\mathbb{C}$-diff'ble only at zero!

6

Panel 7

Good news:

$$\frac{d}{dz} c = 0$$

$$\frac{d}{dz} z = 1$$

$$\frac{d}{dz} z^n = n z^{n-1}$$

$$\frac{d}{dz}\left(f(z) \pm g(z)\right) = f' \pm g'$$

$$\frac{d}{dz}\left(f(z) \cdot g(z)\right) = f g' + f' g$$

$$\frac{d}{dz} \frac{f(z)}{g(z)} = \frac{f' g - f g'}{(g)^2}$$

$$\frac{d}{dz} f(g(z)) = f'(g(z)) \cdot g'(z)$$

Panel 8

Ex: $f(z) = (2z^2 + i)^5$. Find $f'(z)$

$$f'(z) = 5(2z^2 + i)^4 \cdot 4z$$

Bad news: function with $\bar{z}$ are generally not differentiable!

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Panel 9

Cauchy - Riemann Equations:

Suppose $f(z)$ is C-diff'ble at z_0. Then

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = \lim_{(x,y)\to(x_0,y_0)}\frac{u(x,y)+iv(x,y)-\left(u(x_0,y_0)+iv(x_0,y_0)\right)}{(x+iy)-(x_0+iy_0)} =$$

$$= \lim \frac{u(x,y)-u(x_0,y_0)}{x-x_0+i(y-y_0)} + i\,\frac{v(x,y)-v(x_0,y_0)}{x-x_0+i(y-y_0)}$$

these limits exist.

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Panel 10

$$f'(z_0) = \lim_{(x,y)\to(x_0,y_0)}\frac{u(x,y)-u(x_0,y_0)}{x-x_0+i(y-y_0)} + i\,\frac{v(x,y)-v(x_0,y_0)}{x-x_0+i(y-y_0)}$$

Let $y=y_0$, $x\to x_0$: $f'(z_0) = \lim_{x\to x_0}\frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i\lim\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0} =$

$$= \frac{\partial}{\partial x}u + i\frac{\partial}{\partial x}v = u_x + iv_x$$

Let $x=x_0$, $y\to y_0$: $f'(z) = \lim_{y\to y_0}\frac{u(x_0,y)-u(x_0,y_0)}{i(y-y_0)} + i\lim\frac{v(x_0,y)-v(x_0,y_0)}{i(y-y_0)} =$

$$= -i\frac{\partial}{\partial y}u + \frac{\partial}{\partial y}v = v_y - iu_y$$

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Panel 11

We proved that: if $f(z)$ is $\mathbb{C}$-diff'ble, then

$$f'(z) = u_x + i v_x$$
$$= v_y - i u_y$$

$$u_x = v_y$$
$$v_x = -u_y$$

Cauchy-Riemann Equations

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Panel 12

Thm: If $f(z) = u(x,y) + i v(x,y)$ is $\mathbb{C}$-diff'ble at a point, then the Cauchy-Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

must hold. Moreover:

$$f'(z) = u_x + i v_x$$
$$= v_y - i u_y$$

Ex: Is $f(z) = x^2 + i y^3$ $\mathbb{C}$ diff'ble? If yes: $u_x \overset{?}{=} v_y$ $\cancel{2x \overset{?}{=} 3y^2}$

No!

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Panel 13

Ex: Suppose $f(z) = \underbrace{x^2 - y^2}_{u} + \underbrace{2ixy}_{v} = z^2$.

$u_x = 2x \overset{\checkmark}{=} v_y = 2x$ CR are true!

$u_y = -2y \overset{\checkmark}{=} -v_x = 2y$

$f'(z) = u_x + iv_x = 2x + i2y = 2(x+iy) = \underline{2z}$

If you arbitrarily throw together $x^n, y^m, i, \#$, the result will likely <u>NOT</u> be $\mathbb{C}$-diff'ble

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Panel 14

<u>Thm:</u> If $f(z) = u(x,y) + iv(x,y)$ is defined in a nbhd of $z_0 = x_0 + iy_0$, and

(a) u_x, u_y, v_x, v_y exist <u>and</u> are continuous

(b) CR equations hold, i.e. $u_x = v_y$ and $u_y = -v_x$

then f is $\mathbb{C}$-diff'ble!

Read Pf

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Panel 15

Ex: Show that the exp. function $f(z) = e^z$ is diff'ble and find its derivative.

$$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)) = \underbrace{e^x\cos(y)}_{u} + i\underbrace{e^x\sin(y)}_{v}$$

$u_x = e^x\cos(y)$ ✓ $v_y = e^x\cos(y)$

$u_y = -e^x\sin(y)$ ✓ $v_x = e^x\sin(y)$

e^z is diff'ble and $f'(z) = u_x + iv_x = e^x\cos(y) + ie^x\sin(y) =$

$$= e^x(\cos(y) + i\sin(y)) = e^x e^{iy} = e^z$$

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Panel 16

CR equations in Polar Coordinates

Chain rule in $\mathbb{R}$: $\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$

Chain Rule in $\mathbb{R}^2$: $x = x(s,t)$, $y = y(s,t)$ $\Rightarrow$ $f(x,y) = f(s,t)$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

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Panel 17

Ex: $f(x,y) = x y^2$, $x = r\cos(t)$, $y = r\sin(t)$

① $f(t,r) = r\cos(t)\cdot r^2\sin^2(t) = r^3\cos(t)\sin^2(t)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = y^2\cdot\cos(t) + 2xy\sin(t) =$$

$$= r^2\sin^2(t)\cos(t) + 2r^2\sin^2(t)\cos(t) =$$

$$= 3r^2\sin^2(t)\cos(t)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = -y^2 r\sin(t) + 2xy\, r\cos(t) =$$

$$= -r^3\sin^3(t) + 2r^3\sin(t)\cos^2(t)$$

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Panel 18

Suppose $f(z) = u(x,y) + i v(x,y)$ and

CR: $u_x = v_y$

$u_y = -v_x$

If $z = re^{it} \Leftrightarrow x = r\cos(t)$, $y = r\sin(t)$

$$\frac{\partial}{\partial r}u = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x\cos(t) + u_y\sin(t)$$

$$\frac{\partial}{\partial t}u = u_x(-r\sin(t)) + u_y\, r\cos(t)$$

$$\frac{\partial}{\partial r}v = v_x\cos(t) + v_y\sin(t)$$

$$\frac{\partial}{\partial t}v = -v_x r\sin(t) + v_y r\cos(t)$$

HW

$r u_r = v_t$

$u_t = -r v_r$

CR in polar coord.

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Panel 19

Thm (CR in Polar Coordinates)

If f is $\mathbb{C}$-diff'ble and $z = re^{i\theta}$ then

$$r u_r = v_\theta \quad \text{and} \quad u_\theta = -r v_r$$

Conversely, if the partial derivatives are continuous and CR is true then

f is $\mathbb{C}$-diff'ble.

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