Panel 1

A point $z \in D \subset \mathbb{C}$ is called interior point of D if there is an ε-nbhd. of z containing only points of D

A point $z \in \mathbb{C}$ is called exterior point of D if there is an ε-nbhd. of z containing no points from D

A point $z \in \mathbb{C}$ is called a boundary point of D if it is neither interior nor exterior point of D

A set $D \subset \mathbb{C}$ is closed if it contains all of its boundary points.

A point $z_0 \in \mathbb{C}$ is an accumulation point of D if each deleted nbhd. of z_0 contains at least one point of D

1

Panel 2

1. Sketch the following sets and determine which are domains:

(a) $|z - 2 + i| \leq 1$; *(b)* $|2z + 3| > 4$;
(c) $\operatorname{Im} z > 1$; *(d)* $\operatorname{Im} z = 1$;
(e) $0 \leq \arg z \leq \pi/4$ $(z \neq 0)$; *(f)* $|z - 4| \geq |z|$.

2. Which sets in Exercise 1 are neither open nor closed?

3. Which sets in Exercise 1 are bounded?

7. Determine the accumulation points of each of the following sets:

(a) $z_n = i^n$ $(n = 1, 2, \ldots)$; *(b)* $z_n = i^n/n$ $(n = 1, 2, \ldots)$;
(c) $0 \leq \arg z < \pi/2$ $(z \neq 0)$; *(d)* $z_n = (-1)^n(1+i)\dfrac{n-1}{n}$ $(n = 1, 2, \ldots)$.

8. Prove that if a set contains each of its accumulation points, then it must be a closed set.

2

Panel 3

z is accum. point if every ε-nbhd of z contains points in D other than z

D is closed $\Rightarrow$ it contains all bdry points

$z \in \partial D$ if every ε-nbhd of z contains points in D and outside D

Assume D contains all its accumulation points, but D is not closed.

D not closed $\Rightarrow$ D does not contain all its bdry pts.

There is (at least) one $z_0 \in \partial D$ but $z_0 \notin D$

Take any ε-nbhd of z_0 $\Rightarrow$ must contain pts inside ~~and outside~~ different from z_0

$\Rightarrow z_0$ is accumulation point

$\Rightarrow z_0 \in D$ $\quad ↯ \quad$ $\Rightarrow$ Thus, D must be closed

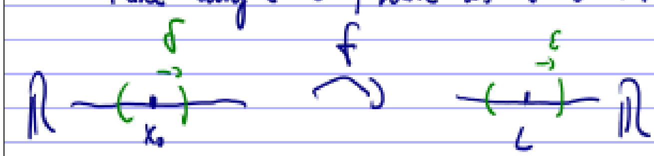
3

Panel 4

Limits and Continuity:

Recall: $\lim_{x \to x_0} f(x) = L$ means:

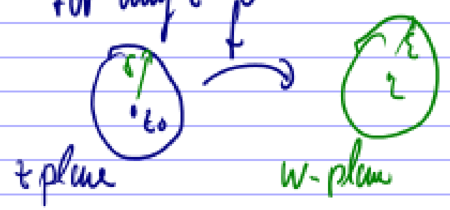
Take any $\varepsilon > 0$, there is $\delta > 0$ s.t. if $|x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$



In $\mathbb{C}$: $\lim_{z \to z_0} f(z) = L$ means

for any $\varepsilon > 0$ there is $\delta > 0$ s.t. if $|z - z_0| < \delta$

$\Rightarrow |\underbrace{f(z)}_{w} - L| < \varepsilon$

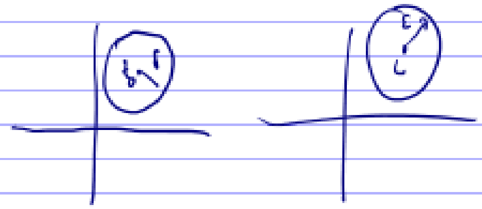


4

Panel 5

The Trouble with Limits in $\mathbb{C}$: $\lim_{z \to z_0} f(z) = L$ means

$$|z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$$



There are many ways to be close to z_0, not only left/right as in $\mathbb{R}$

$$f(x,y) = \frac{xy}{x^2+y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} =$$

Let $x=0$: $\lim \frac{0y}{0^2+y^2} = \frac{0}{y^2} = 0$

Let $y=0$: $\lim = 0$

Let $x=y$: $\lim \frac{x^2}{x^2+x^2} = \frac{1}{2}$

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$\mathbb{R}$: $\lim_{x \to 0} \frac{x^2}{\sin(x)} = \frac{2x}{\cos(x)} = 0$

Panel 6

Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be a complex function defined in a nbhd. of $z_0 = (x_0, y_0)$. Then

$$\lim_{z \to z_0} f(z_0) = w_0 = u_0 + iv_0$$

iff: $\lim u(x,y) = u_0$ and $\lim v(x,y) = v_0$

Thus, limits in $\mathbb{C}$ are exactly like those in $\mathbb{R}^2$

Panel 7

Ex: $f(z) = z/\bar{z}$. Then $\lim_{z \to 0} f(z)$ does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x+iy}{x-iy}:$$

let $x=0$: $\lim \frac{iy}{-iy} = -1$

let $y=0$: $\lim \frac{x}{x} = 1$

different!

7

Panel 8

Def: f a complex function defined in a nbhd of z_0. Then f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

same as $\mathbb{R}^2$.

8

Panel 9

Continuity Theorems

- A polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ is cont.
- A rational function $r(z) = p_n(z) / q_m(z)$ is cont. except where $q_m(z) = 0$
- The sum of two continuous functions is cont.
- The product of two continuous functions is cont.
- The quotient of two continuous functions is cont. except where denom $= 0$

9

Panel 10

Ex: Find $\lim\limits_{z \to 1+i} \frac{z^2 - 2i}{z^2 - 2z + 2}$ $\quad \frac{(1+i)^2 - 2i}{(1+i)^2 - 2(1+i) + 2} = \frac{0}{0}$

$$z^2 - 2i = (z + \sqrt{2i})(z - \sqrt{2i}) = (z + (1+i))(z - (1+i))$$

$$z^2 - 2z + 2 = 0 \qquad z = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm 2i}{2} = 1+i,\ 1-i$$

$$\lim_{z \to (1+i)} \frac{z^2 - 2i}{z^2 - 2z + 2} = \lim_{z \to (1+i)} \frac{(z + (1+i))\cancel{(z - (1+i))}}{\cancel{(z - (1+i))}(z - (1-i))} = \frac{1+i+1+i}{1+i-1+i} = \frac{2+2i}{2i}$$

$$\frac{1+i}{i} = \frac{-i+1}{1} = 1 - i$$

10