Panel 1

Last Time: Root Day

Of $z^n = a = re^{i\theta}$ always has n different solutions

Algebraically: $z_k = (re^{i\theta})^{1/n} = r^{1/n} e^{i(\frac{\theta}{n} + \frac{2k\pi}{n})}$, $k=0,1,\dots,n-1$

Geometrically: n n-th roots are symmetrically distributed and form sides of a regular polygon with n-sides, the first angle is θ/n

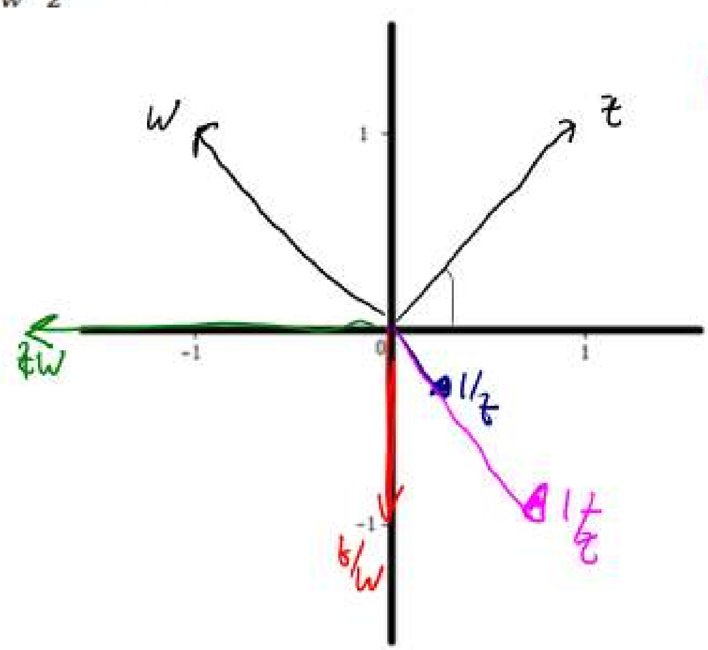
Roots of Unity: $z^n = 1$ $\quad w_k = e^{i\frac{2k\pi}{n}}$, $k=0,1,\dots,n-1$

1

Panel 2

Consider $z = 1 + i$ and $w = -1 + i$. Draw the vectors $z, w, z \cdot w, \frac{z}{w}, \frac{1}{z}$, and $\bar{z}$

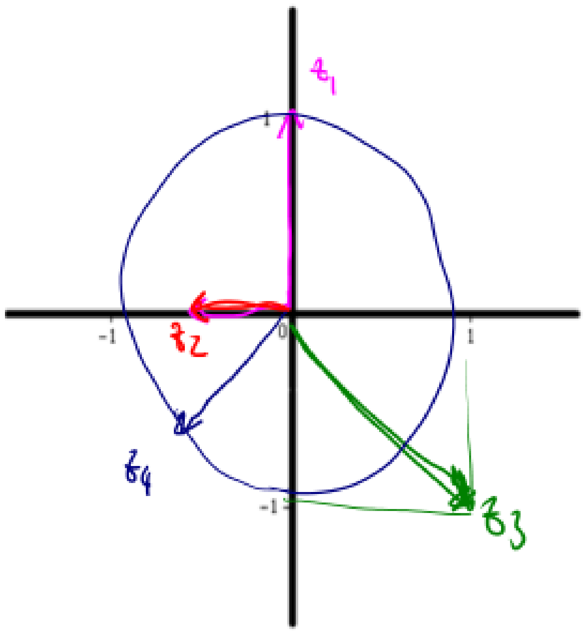
$\frac{1}{re^{i\theta}} = \frac{1}{r} e^{i(-\theta)}$



2

Panel 3

Draw the following vectors: $z_1 = e^{\frac{i\pi}{2}}$, $z_2 = 0.5e^{i\pi}$, $z_3 = \sqrt{2}e^{\frac{-i\pi}{4}}$, and $z_4 = e^{\frac{i5\pi}{4}}$



3

Panel 4

Describe in simple geometric terms what happens to a vector z when:

a. it is multiplied by 2

2× as long

b. it is multiplied by -1 opposite direction

c. it is multiplied by i rotates by 90°

d. it is squared length is squared, angle doubles

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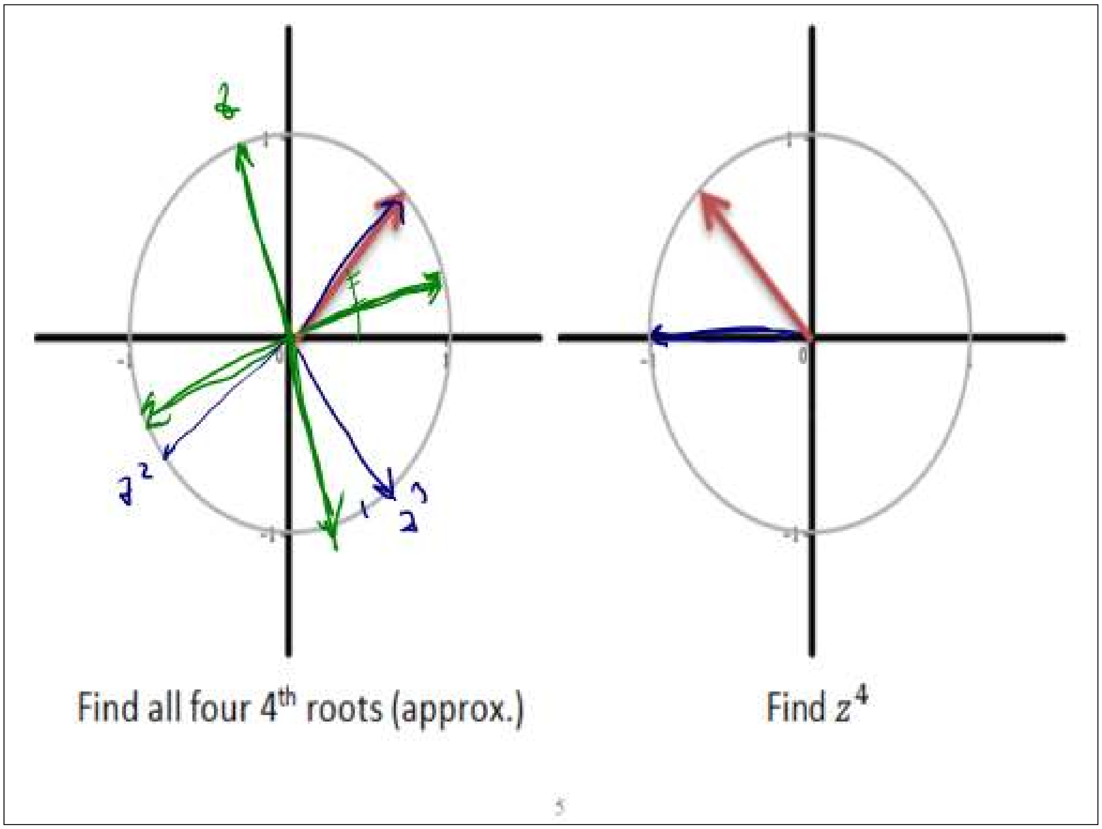
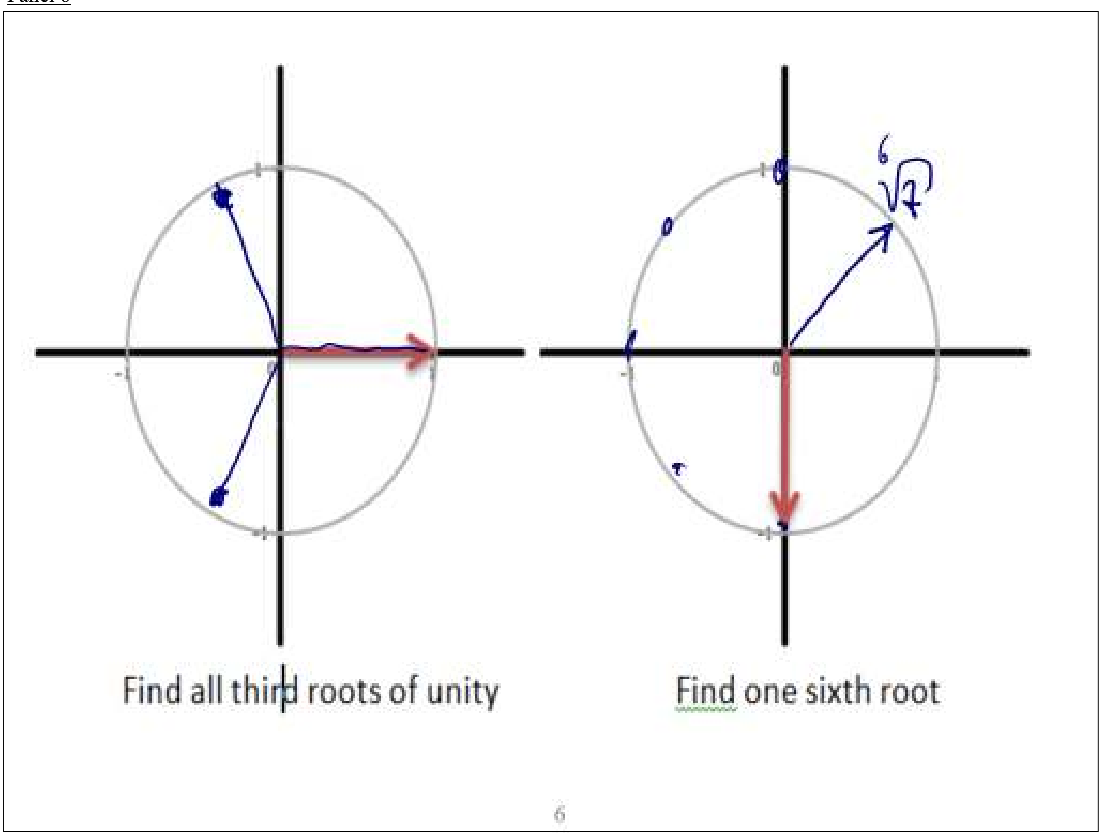
Panel 5



Panel 6



Panel 7

$(e^{it})^3 = e^{i3t}$

$|z| = 1$

$(\cos(t) + i\sin(t))^3 = \cos(3t) + i\sin(3t)$

$z = 1, \ z = -1$

$(\cos t + i\sin(t))(\cos(t) + i\sin t)^2$

$|x| = 1$

$(\cos t + i\sin(t))(\cos^2(t) - \sin^2(t) + 2i\cos(t)\sin(t))$

$\cos^3(t) - \cos(t)\sin^2(t) + 2i^2\cos(t)\sin^2(t) + i(\quad)$

$\cos^3 - 3\cos\sin^2 \quad + i(\quad)$

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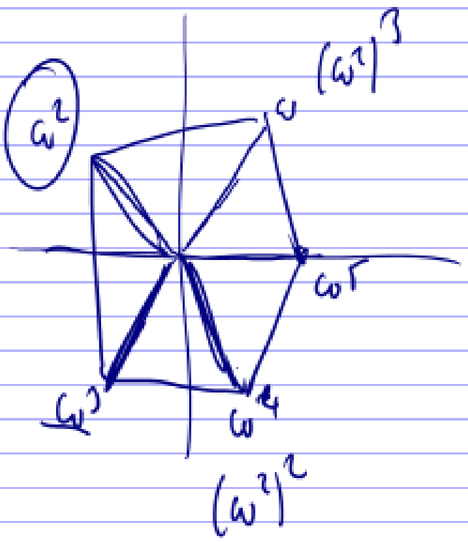
Panel 8

$\omega_n = e^{i\frac{2k\pi}{n}}$



$\omega_0 + \omega_1 + \omega_2 + \dots + \omega_{n-1} =$

$1 + e^{i\frac{2\pi}{n}} + e^{i\frac{4\pi}{n}} + e^{i\frac{6\pi}{n}} + \dots + e^{i\frac{(n-1)\pi}{n}} =$

$1 + e^{i\frac{2\pi}{n}} + \left(e^{i\frac{2\pi}{n}}\right)^2 + \left(e^{i\frac{2\pi}{n}}\right)^3 + \left(e^{i\frac{2\pi}{n}}\right)^{n-1} =$

$\frac{1 - \left(e^{i\frac{2\pi}{n}}\right)^n}{1 - e^{i\frac{2\pi}{n}}} = \frac{1-1}{\#} = 0$

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Panel 9

Lemma: $1 + z + z^2 + \dots + z^{n-1} = \dfrac{1-z^n}{1-z}$

$$S = 1 + \cancel{z} + \cancel{z^2} + \dots + \cancel{z^{n-1}} \quad \Big| \cdot z$$

$$- \quad zS = \cancel{z} + \cancel{z^2} + \dots \cancel{\ } + z^n$$

$$S - zS = 1 - z^n$$

$$S(1-z) = 1 - z^n \quad \Rightarrow \quad \underline{S = \frac{1-z^n}{1-z}}$$

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Panel 10

Complex Functions

A complex function is a rule that assigns to every z in a domain $D \subset \mathbb{C}$ a complex number w.

Ex: $f(z) = z^2$ $\qquad$ $g(z) = x^2 + y^2$ $\qquad$ $D = \mathbb{C}$

$h(z) = \dfrac{1-z}{1+z\bar{z}}$ $\qquad$ $k(z) = z - \bar{z}$

Note: If no domain is specified explicitly, we assume the largest possible subset of $\mathbb{C}$ as domain.

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Panel 11

Thm: Every complex function $f(z) = w$ can be written as $f(z) = u(x,y) + i\,v(x,y)$

Ex: $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$, $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$

$g(z) = z\bar{z} = x^2 + y^2$, $u(x,y) = x^2 + y^2$, $v(x,y) = 0$

$h(z) = ix^2$ $u(x,y) = 0$, $v(x,y) = x^2$

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Panel 12

Thm: Every function $u(x,y) + iv(x,y)$ can be converted to a function $f(z,\bar{z})$

Proof. $x = \frac{z+\bar{z}}{2}$ $y = \frac{z-\bar{z}}{2i}$

Ex: $u(x,y) + iv(x,y) = x^2 + iy^2 = \left(\frac{z+\bar{z}}{2}\right)^2 + i\left(\frac{z-\bar{z}}{2i}\right)^2 =$

$= \frac{1}{4}\left(z^2 + 2z\bar{z} + \bar{z}^2\right) - \frac{i}{4}\left(z^2 - 2z\bar{z} + \bar{z}^2\right) -$

$u(x,y) + iv(x,y) = x^2 - y^2 + 2ixy = \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 + 2i\,\frac{z+\bar{z}}{2}\cdot\frac{z-\bar{z}}{2i} =$

HW

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Panel 13

Graphs of Complex Functions

$f: \mathbb{R} \to \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$

$f: \mathbb{C} \to \mathbb{C}$, $f: \mathbb{R}^2 \to \mathbb{R}^2$

graph of a complex function is a 4D object. That is too bad!

Good news: analyze $f: \mathbb{C} \to \mathbb{C}$ to try to understand 4D!

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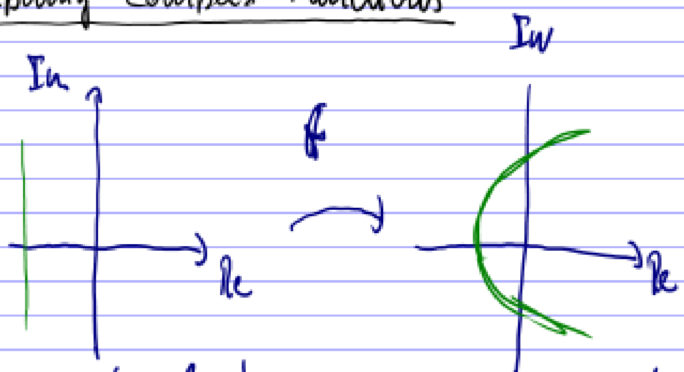
Panel 14

Graphing Complex Functions



domain (z-plane) range (w-plane)

f transforms z-plane into the w-plane

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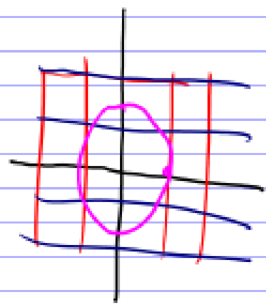
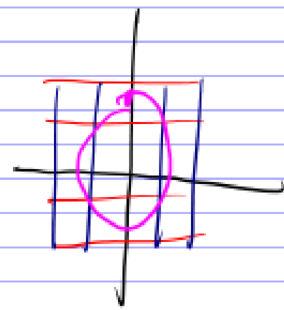
Panel 15

Ex: Describe the mapping properties of mult. by i, i.e.

$$f(z) = iz = e^{i\pi/2} \cdot z = e^{i\pi/2}\, r e^{it} = r e^{i(t+\pi/2)}$$



z-plane



w-plane

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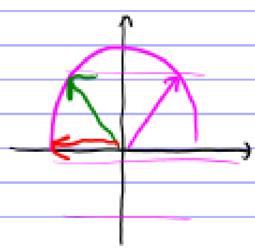
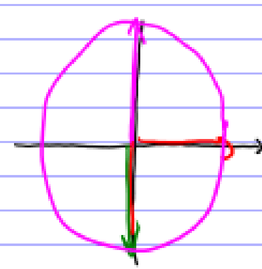
Panel 16

Ex: What does $f(z) = z^2$ do to circles and radii?





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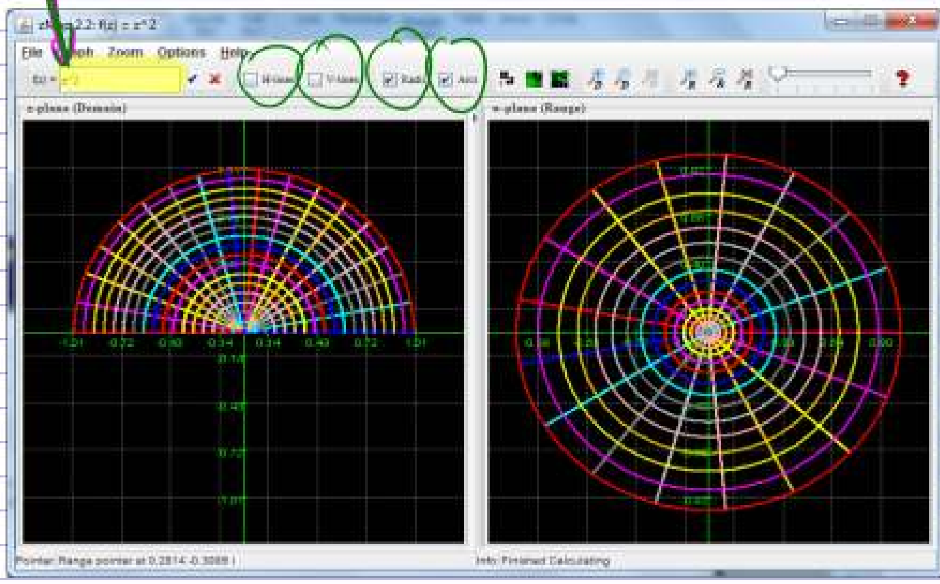
Panel 17

zMap: Program to visualize complex functions

f(z) =



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Panel 18

http://www.mathcs.org/java/programs/ZMap/index.html

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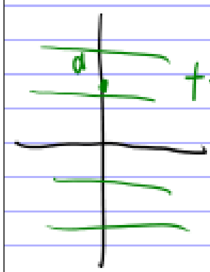
Panel 19

$f(z) = z^2$

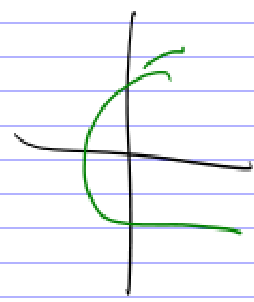
$= (t+ia)^2 = \underbrace{t^2 - a^2}_{x} + \underbrace{2iat}_{y}$

$a=1: \; x = \frac{y^2}{4} - 1$

$\underline{x = t^2 - a^2}, \; 2at = y \Rightarrow t = \frac{y}{2a} \Rightarrow x = \left(\frac{y}{2a}\right)^2 - a^2$



$t + ia, \; t \in [-1,1]$



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Panel 20

Graph of $e^z = e^{x+iy} = \underline{e^x e^{iy}}$

<u>fix x:</u> $\Rightarrow e^x e^{iy}$ is fixed radius, circle as y varies

$\Rightarrow$ vert lines $\rightarrow$ circles

<u>fix y:</u> $\rightarrow$ horiz lines $\rightarrow$ radii with fixed angles

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