Panel 1

Last Time:

$z \in \mathbb{C}$ means $z = (x, y) = x + iy$

$z_1 + z_2$

$z_1 \cdot z_2$

$Re(z)$, $Im(z)$

z/w

equations: $z^2 = i$ or $\sqrt{i}$

1

Panel 2

$z = 3 + 4i$

$z = x + iy$

$z^2 = x^2 - y^2 + 2xyi$

$\begin{cases} x^2 - y^2 = 3 \\ 2xy = 4 \Rightarrow xy = 2 \end{cases}$

$\begin{cases} x^2 - y^2 = 3 \\ xy = 2 \end{cases} \Rightarrow y = \frac{2}{x}$

$x^2 - \left(\frac{4}{x^2}\right) = 3 \Rightarrow x^4 - 4 - 3x^2 = 0$

$x^2 = u \Rightarrow u^2 - 3u - 4 = 0$

$(u + 1)(u - 4) = 0$

$u = -1 \Rightarrow x^2 = -1$

No Solutions

$u = 4 \Rightarrow x^2 = 4$

$x = \pm 2$

$y = \frac{2}{x}$

$x = 2,\ y = 1$

$x = -2,\ y = -1$

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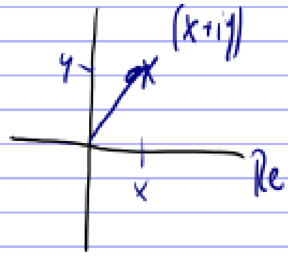
Panel 3

Complex Numbers Graphically

$z = (x,y) = x + iy$:

In



is a vector in $\mathbb{R}^2 \cong \mathbb{C}$

Def: $\|z\| = |z| = \sqrt{x^2+y^2}$

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Panel 4

Quick Properties:

If $c \in \mathbb{R}$ and $z \in \mathbb{C}$ then $|c \cdot z| = |c|\,|z|$

If $z, w \in \mathbb{C}$ then $|z \cdot w| = |z| \cdot |w|$ ← Prove

$Re(z) \le |Re(z)| \le |z| = \sqrt{x^2+y^2}$

$Im(z) \le |Im(z)| \le |z| = \sqrt{x^2+y^2}$

Triangle Inequality:

$|z+w| \le |z| + |w|$

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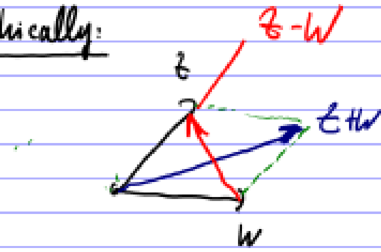
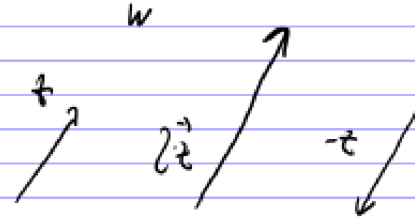
Panel 5

Complex Numbers Graphically:

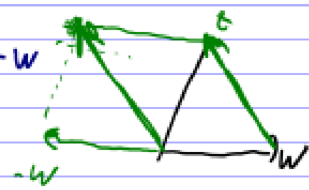
How to add: $z+w$



How to find $c \cdot z$, $c \in \mathbb{R}$



How to subtract: $z-w$



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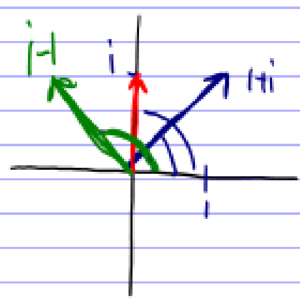
Panel 6

How to multiply: $z \cdot w$

$(1+i) \cdot i = i - 1$

? Graphically



$|(1+i)(1+i)| = |1+i||1+i| = (\sqrt{2})^2 = 2$

Don't know yet.

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Panel 7

Complex Conjugate

Def: If $z = x + iy$, then the conjugate of z is:

$$\bar{z} = x - iy$$

Ex: $\overline{3+4i} = 3-4i$

$\overline{-1-i} = -1+i$

$\overline{(1+i)(1-i)} = (1-i)(1+i)$ ← Proof

$\frac{z+\bar{z}}{2} = \frac{x+iy+(x-iy)}{2} = x = Re(z)$

$\frac{z-\bar{z}}{2i} = \frac{x+iy-(x-iy)}{2i} = y = Im(z)$

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Panel 8

Properties of Conjugates

(a) $\overline{z+w} = \bar{z} + \bar{w}$

(b) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

(c) $\overline{\left(\frac{z}{w}\right)} = \bar{z}/\bar{w}$

(d) $\overline{(\bar{z})} = z$

! (e) $Re(z) = \frac{z+\bar{z}}{2}$

! (f) $Im(z) = \frac{z-\bar{z}}{2i}$

! (g) $z \cdot \bar{z} = (x+iy)(x-iy) = x^2+y^2 = \left(\sqrt{x^2+y^2}\right)^2 = \|z\|^2$

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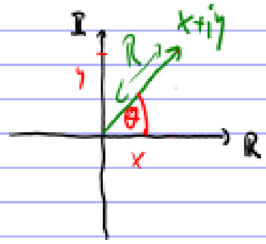
Panel 9

Still not solved: How to visualize complex multiplication



Can write $z = x+iy = (x,y)$

$z = R(\cos\theta, \sin(\theta)) =$

$= R(\cos\theta + i\sin\theta)$, $R = \|z\|$

Def: Every complex number z can be written as

$z = r(\cos\theta + i\sin\theta)$, $r = \|z\| = \sqrt{x^2+y^2}$.

Def: $\theta = \arg(z)$ (Argument)

$\mathrm{Arg}(z) = \theta$, $-\pi < \mathrm{Arg}(z) \le \pi$, Principal Argument

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Panel 10

Ex: $z = 5 \Rightarrow \arg(z) = 0, 2\pi, 4\pi, -2\pi$

$\mathrm{Arg}(z) = 0$

$z = i \Rightarrow \arg(z) = \pi/2, \ \pi/2 \pm 2k\pi$

$\mathrm{Arg}(z) = \pi/2$

$z = -i \Rightarrow \arg(z) = \frac{3\pi}{2}, -\pi/2, \ldots$

$\mathrm{Arg}(z) = -\pi/2$

$z = -5 \Rightarrow \arg(z) = \pi, 3\pi, -\pi, \ldots$

$\mathrm{Arg}(z) = \pi$

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Panel 11

Recall from Calc 2:

$\Rightarrow e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

Thus: $e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \dots$

$1 + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \dots$

$\cos(x) + i\sin(x)$

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Panel 12

Euler's Formula: $e^{it} = \cos(t) + i\sin(t)$

Thus: Take any $z \in \mathbb{C}$, $z \neq 0$. Then

$z = x + iy = r(\cos(t) + i\sin(t)) = re^{it}$

Ex: $e^{i\pi/2} = i$, $\|e^{it}\| = \|\cos t + i\sin(t)\| = \sqrt{\cos^2(t) + \sin^2(t)} = 1$

$e^{i\pi} = -1$

Ex: $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 \iff \boxed{e^{i\pi} + 1 = 0}$ way cool

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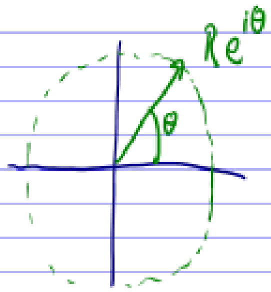
Panel 13

Geometrically: $z = Re^{i\theta}$ is the a circle, radius R, center 0



$\|z\| = R$

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Panel 14

How does this help visualizing multiplication?

$z_1 = R_1 e^{it_1}, \quad z_2 = R_2 e^{it_2}$

$$z_1 \cdot z_2 = R_1 e^{it_1} R_2 e^{it_2} = R_1 R_2 (\cos(t_1) + i\sin(t_1))(\cos(t_2) + i\sin(t_2)) =$$

$$= R_1 R_2 (\cos(t_1)\cos(t_2) - \sin(t_1)\sin(t_2) + i(\cos(t_1)\sin(t_2) + \cos(t_2)\sin(t_1)))$$

$$= R_1 R_2 (\cos(t_1 + t_2) + i\sin(t_1 + t_2))$$

$$= R_1 R_2 e^{i(t_1 + t_2)}$$

$$R_1 e^{it_1} \cdot R_2 e^{it_2} = R_1 R_2 e^{i(t_1 + t_2)}$$

mult. lengths

add angles

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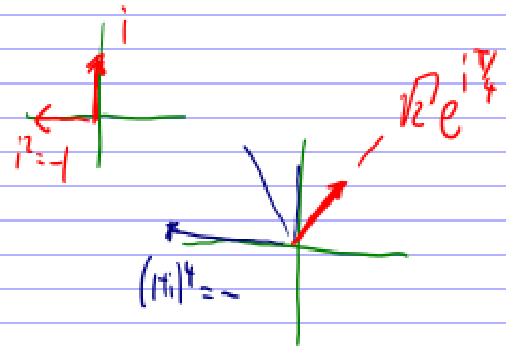
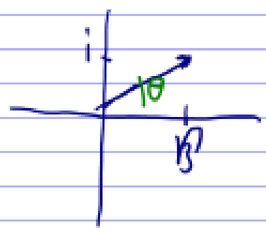
Panel 15

Ex: $i^2 = i \cdot i = -1$



$(1+i)^4 = -(\sqrt{2})^4$

$(\sqrt{3}+i)^7$



$\sqrt{3}+i = 2e^{i\frac{\pi}{6}}$

$(\sqrt{3}+i)^7 = 2^7 e^{i\frac{7\pi}{6}} = 2^7 e^{i\pi} e^{i\frac{\pi}{6}}$

$= -2^7(\sqrt{3}+i)$

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Panel 16

Note: $(e^{it})^n = e^{int}$

De Moivre's Formula: $(\cos(t)+i\sin(t))^n = \cos(nt)+i\sin(nt)$

n=2: $(\cos(t)+i\sin(t))^2 = \cos(2t)+i\sin(2t)$

$\cos^2(t)-\sin^2(t)+2i\sin(t)\cos(t) = ?$

$\cos^2(t)-\sin^2(t) = \cos(2t)$

$2\sin(t)\cos(t) = \sin(2t)$

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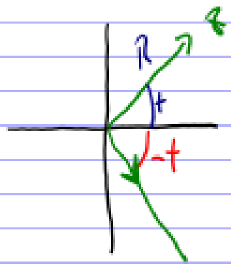
Panel 17

Visualize Division



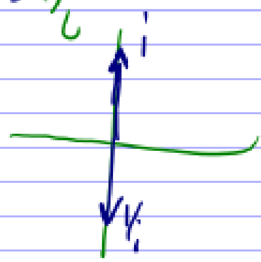
$$1/z = \left(\frac{1}{Re^{it}}\right) = \frac{1}{R} e^{-it} =$$

$$= \frac{1}{R}\left(\cos(-t) + i\sin(-t)\right) =$$

$$= \frac{1}{R}\left(\cos(t) - i\sin(t)\right) =$$

$$= \frac{1}{R}$$

$-i = 1/i$



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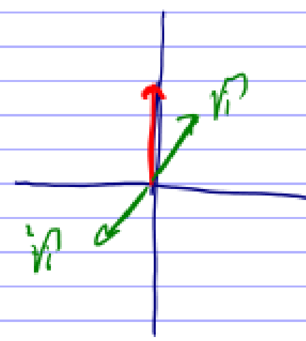
Panel 18

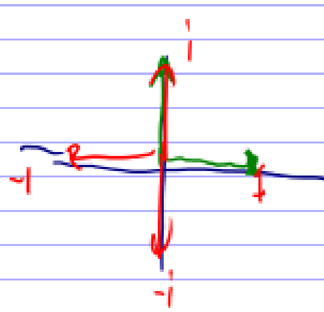
Complex Roots

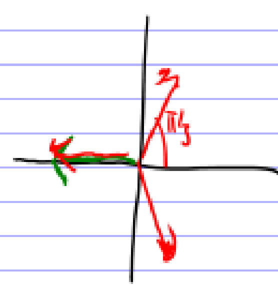
$\sqrt[2]{i}$

$\sqrt[4]{1}$

$\sqrt[3]{-1}$







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