Complex Practice Exam 1

*This practice exam contains **sample** questions. The actual exam will have fewer questions, and may contain questions not listed here.*

1. Be prepared to explain the following concepts, definitions, or theorems:
   - A complex number, polar coordinates, rectangular coordinates
   - Add, Multiply, Sub, Div, Conjugate, abs Value, graphical interpretations of these
   - Complex roots
   - Mapping properties of complex functions
   - Arg(z) and arg(z)
   - The limit of a complex function f(z) as z approaches c is L
   - Continuity of a complex function f(z) at a point z = c
   - The complex derivative of a function f(z)
   - Analytic function and Entire function
   - CR equations
   - f(z) analytic & f'(z) = 0, f(z) analytic & f-conjugate analytic, f(z) analytic and |f(z)| constant
   - Harmonic function and harmonic conjugate of a function u (incl. how to find)
   - e^z, sin(z), cos(z), log(z), and Log(z)
   - Euler's Formula, De Moivre's Formula
   - Complex parametric functions z(t), their integrals and derivatives
   - Different paths (line segments and circles)
   - Contour Integrals

2. Describe the set of points z such that (a) $\mathrm{Re}(z) = 1$ (b) $|z - 1| = 2$ (c) $Arg(z) = \frac{\pi}{4}$

3. Let $z = 1 + i$. Draw, in one coordinate system, $\bar{z}$, $\frac{1}{z}$, z^3, and $z^{\frac{1}{4}}$

4. Compute/simplify the following and find real and imag parts:

   a) $\left|(\overline{1+i})(1-i)i\right|$ (b) $\frac{i(1+i)^3}{(1-i)^2}$ (c) $(1+i)^6$ (d) $\frac{2+2i}{-\sqrt{3}+i}$

5. Find the fourth roots of -1, i.e. $\sqrt[4]{-1}$, and display them graphically. Do the same for the fifth roots of -1 and of (1+i).

6. Find the image of the line $y = 2x$ under the map $f(z) = iz - 1$. What is the image of the unit circle under the same map? Make sure to represent your answers algebraically as well as graphically.

7. Consider the following questions, involving limits and continuity of complex functions. Remember that limits can be taken in different directions, and for complicated limits there is l'Hospital's rule.

   a) If $f(z)=\frac{x-iy}{x+iy}$, then f is clearly undefined at $z = 0$. Can you define *f(0)* in such a way that the new function is continuous at every point in the complex plane?

   b) Say $f(z)=\frac{z^9+z-2i}{z^{15}+i}$ Can you define *f(i)* in such a way that the new function is continuous at every point in the complex plane?

   c) Find $\lim_{z\to 1}\frac{1+z^6}{1+z^{10}}$, $\lim_{z\to i}\frac{1+z^6}{1-z^{10}}$, and $\lim_{z\to i}\frac{1+z^6}{1+z^{10}}$

8. Consider the following questions about analytic functions.

   a) If $f(z)=\frac{1}{\left(z^2+1\right)^2}$ then determine where, if at all, the function is analytic. If it is analytic, find the complex derivative of *f*.

   b) If $f(z)=x^3-3xy^2+i\left(3x^2y-y^3\right)$ then determine where, if at all, the function is analytic. If it is analytic, find the complex derivative of *f*.

9. Decide which of the following functions are analytic, and in which domain they are analytic. If a function is analytic, find its complex derivative:

   (a) $f(z)=\frac{e^z+1}{e^z-1}$ (b) $f(z)=x^3+3ix^2y-3xy^2+x-iy^3+iy$

10. Consider the function $u(x,y)=e^x\sin(y)$. Is it harmonic ? If so, find its harmonic conjugate. Do the same for (a) $u(x,y)=x^3-2xy+xy^3$ (b) $u(x,y)=e^y\cos(x)$

11. Please find the following numerical answers:

    (a) e^{2+2i}, (b) $\cos(\pi+i)$, (c) $\sin\left(i-\frac{\pi}{2}\right)$

14 Solve the following equations for z.

(a) $z^4+1=0$, (b) $\left|e^{2z}\right|=3$, (c) $\sin(z)=3i$, (d) $e^{4z}=1$, (e) $\cos(z)=i\sin(z)$

15 Use the definition of derivative to show that the functions $f(z)=\mathrm{Re}(z)$ is nowhere differentiable. Use the CR equations to show that the function $f(z)=\bar{z}$ is nowhere differentiable. Show that if *v* is the harmonic conjugate of *u*, then the product *u v* is harmonic.

16 Show that $\left|e^z\right|\le 1$ if $\mathrm{Re}(z)\le 0$

17 State De Moivre's formula. Then use it to prove the trig identity $\sin(2x) = 2\sin(x)\cos(x)$

18 Show that the function e^{iz} is periodic with period 2π

19 Show that the function *sin(z)* is unbounded

20 Show that the function $f(z) = z\bar{z} + z + \bar{z} + 2x$ can not be an analytic function.

21 Prove that $\sin^2(z) + \cos^2(z) = 1$ (Hint: take the derivative of $f(z) = \sin^2(z) + \cos^2(z)$)

22 Prove the following theorem: If *f(z)* is an analytic function with values that are always imaginary, then the function must be constant.

23 Prove the following theorem: if is a harmonic function in an open set U (i.e. h is twice continuously differentiable and $\frac{\partial^2}{\partial x^2}h + \frac{\partial^2}{\partial y^2}h = 0$ in the open set U), then the complex function $f(z) = \frac{\partial}{\partial x}h(x,y) - i\frac{\partial}{\partial y}h(x,y)$ is an analytic function in U.

24 Find complex parametric functions representing the following paths: (a) a straight line from –i to i, (b) the right half of a circle from –i to i, (c) a straight line from -1 – 2i to 3 + 2i (d) a circle centered at 1+i of radius 2 – d

25 Evaluate

a. z'(t) for $z(t) = \cos(2t) + i\sin(2t)$

b. $\int_0^{\pi} z(t)dt$ for $z(t) = (5+4i)e^{3it}$

26 Evaluate

a. $\int_{\gamma} iz^2 + 3dz$ where γ is a line segment from -1-i to 1+i

b. $\int_{\gamma} \frac{1}{\bar{z}}dz$ where γ is a circle radius 2 centered at the origin