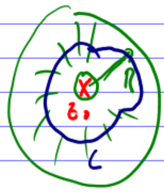
Panel 1

Last Time:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad 0<|z-z_0|<R$$

Special member: $a_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz$



Def: $a_{-1} = \mathrm{Res}(f, z_0)$

Note: to find $\mathrm{Res}(f, z_0)$, you must write f in powers of $(z-z_0)^n$

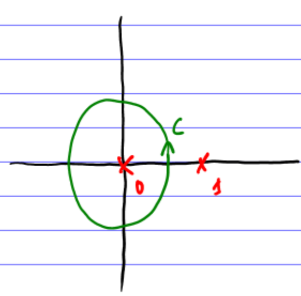
1

Panel 2

$f(z) = \frac{2}{z(z-1)}$ $\qquad \mathrm{Res}(f,0) = -2$ and $\mathrm{Res}(f,1) = 2$

Find $\int_C \frac{2}{z(z-1)}\,dz = 2\pi i\,\mathrm{Res}(f,0) = -4\pi i$

a) C is circle center $z_0 = 0$, radius $1/2$



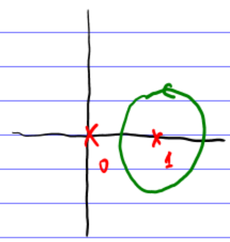
2

Panel 3

$f(z) = \frac{2}{z(z-1)}$ $\quad$ $\mathrm{Res}(f,0) = -2$ and $\mathrm{Res}(f,1) = 2$

Find $\displaystyle\int_C \frac{2}{z(z-1)}\,dz = 2\pi i\,\mathrm{Res}(f,1) = \underline{\underline{4\pi i}}$

b) C is circle center $z_0 = 1$, radius $1/2$



Note: $\displaystyle\int_C f(z)\,dz = -\int_{-C} f(z)\,dz$

reverse orientation

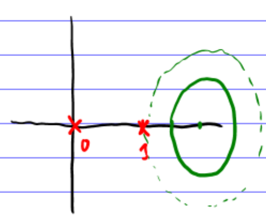
3

Panel 4

$f(z) = \frac{2}{z(z-1)}$ $\quad$ $\mathrm{Res}(f,0) = -2$ and $\mathrm{Res}(f,1) = 2$

Find $\displaystyle\int_C \frac{2}{z(z-1)}\,dz = 2\pi i\,\mathrm{Res}(f,2) = 0$

c) C is circle center $z_0 = 2$, radius $1/2$



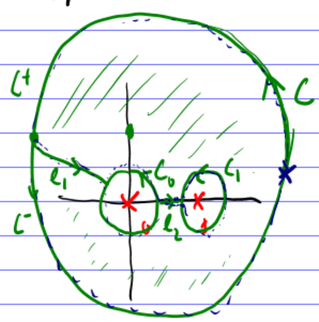
$\displaystyle\frac{2}{z(z-1)} = \sum_{n=0}^{\infty} a_n (z-2)^n, \quad |z-2| < 1$

4

Panel 5

Find $\int_C \frac{2}{z(z-1)}\,dz$ if

a) C is circle center $z_0 = i$, radius 10



$0 = \int f(z)\,dz =$

$C^+ + \ell_1 - C_0^+ + \ell_2 - C_1 - \ell_2 - C_0^- - \ell_1 + C^-$

$= \int_{C - C_0 - C_1} f(z)\,dz = 0$

$\Rightarrow \int_C f(z)\,dz = \int_{C_0} f(z)\,dz + \int_{C_1} f(z)\,dz$

$= 2\pi i \left(\text{Res}(f,0) + \text{Res}(f,1)\right)$

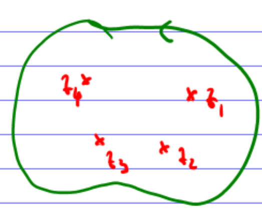
5

Panel 6

The Residue Theorem

Hi-lite of the course OMG!

Suppose f is analytic in a domain D except for finitely many isolated singularities $z_1, z_2, z_3, \ldots, z_n$. If C is a simple curve in D, positively oriented, such that $z_1, z_2, \ldots, z_n$ are inside C, then



$$\int_C f(z)\,dz = 2\pi i \left(\sum_{j=1}^{n} \text{Res}(f, z_j)\right) =$$

$$= 2\pi i \left(\text{Res}(f, z_1) + \ldots + \text{Res}(f, z_n)\right)$$

6

Panel 7

HW Find the residues for the functions as given:

a) $\text{Res}(f,0)$, $f(z) = z^5 \cos(1/z) = z^5\left(1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \dots\right)$

b) $\text{Res}(f,0)$, $f(z) = z^2 e^{1/z} = z^2\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots\right)$

$1/3!$

c) $\text{Res}(f,0)$, $f(z) = \frac{3}{z(z+2)}$ work

d) $\text{Res}(f,-2)$, $f(z) = \frac{3}{z(z+2)}$ work

e) $\text{Res}(f,3)$, $f(z) = \frac{3}{z(z+2)}$ $\text{Res}(f,3) = 0$

f) $\text{Res}(f,0)$, $f(z) = \frac{1}{z^3(z-2)}$ work

7

Panel 8

need powers of z!

$f(z) = \frac{3}{z(z+2)}$, $\text{Res}(f,0) = 3/2$

$$= \frac{3}{z} \cdot \frac{1}{2+z} = \frac{3}{2z} \cdot \frac{1}{1+z/2} = \frac{3}{2z} \sum_{n=0}^{\infty} (-z/2)^n =$$

$$= \frac{3}{2z}\left(1 - z/2 + z^2/4 - z^3/8 + \dots\right) \Rightarrow$$

$f(z) = \frac{3}{z(z+2)}$ $\text{Res}(f,-2) \to$ need powers of $(z+2)$

$$= \frac{3}{z+2} \cdot \frac{1}{-2+z+2} = \frac{3}{z+2} \cdot \frac{1}{(-2)} \cdot \frac{1}{1 - \frac{z+2}{2}} = \frac{3}{-2(z+2)} \sum_{n=0}^{\infty} \left(\frac{z+2}{2}\right)^n =$$

$$= \frac{-3}{2(z+2)}\left(1 + \frac{z+2}{2} + \frac{(z+2)^2}{4} + \dots\right)$$

$\text{Res}(f,2) = -3/2$

8

Panel 9

f) $\text{Res}(f, 0)$, $f(z) = \frac{1}{z^3(z-2)}$

↑ powers of z

$$\frac{1}{z^3(z-2)} = \frac{1}{z^3} \cdot \frac{1}{z-2} = \frac{1}{z^3}\frac{1}{(-2)}\frac{1}{1-z/2} = -\frac{1}{2z^3}\sum_{n=0}^{\infty}(z/2)^n =$$

$$= -\frac{1}{2z^3}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \frac{z^4}{2^4} + \dots\right)$$

$$= -\frac{1}{2z^3}\sum_{n=0}^{\infty} z^n/2^n = -\sum_{n=0}^{\infty}\frac{z^{n-3}}{2^{n+1}}$$

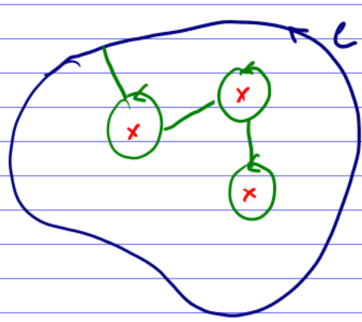
looking for z^{-1}, i.e. $n=2$ $\Rightarrow -1/2^{2+1} = -1/8$

9

Panel 10

Residue Thm: If f is analytic inside and on a simple curve C except finitely many isolated singularities $z_1, z_2, \dots, z_n$, or z_j, $j=1,2,\dots n$.

then



$$\int_C f(z)\,dz = 2\pi i\left(\sum_{j=1}^{n}\text{Res}(f, z_j)\right)$$

Proof: Whip out scissors and cut along the lines.

Apply Cauchy-Goursat

+ Laurent series thm for each circle

10

Panel 11

Ex: $\int_C \frac{5z-2}{z(z-1)}\,dz$, C is circle, radius 2 center at 0, pos. oriented

$= 2\pi i\left(\text{Res}(f,0) + \text{Res}(f,1)\right) = 2\pi i\,(2+3) = \underline{\underline{10\pi i}}$

$\text{Res}(f,0)$: $\frac{5z-2}{z(z-1)} = \frac{5z-2}{z}\cdot\frac{-1}{1-z} =$

z

$-\left(5-\frac{2}{z}\right)\left(1+z+z^2+z^3+\dots\right) \Rightarrow \text{Res}(f,0) = +2$

$\text{Res}(f,1) = \frac{5z-2}{z(z-1)} = \frac{5z-2}{z-1}\cdot\frac{1}{1+(z-1)} =$

Goal: $(z-1)^n$

$= \frac{5(z-1)+3}{z-1}\left(1-(z-1)+(z-1)^2-\dots\right)$

$= \left(5+\frac{3}{(z-1)}\right)\left(1-\dots\right) \Rightarrow \text{Res}(f,1) = 3$

11

Panel 12

Goal: Finding Residues by classifying singularities.

If f is not analytic at z_0 but is analytic $0<|z-z_0|<R$. Then z_0 is isolated singularity. Moreover

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\frac{a_{-1}}{(z-z_0)} + \frac{a_{-2}}{(z-z_0)^2} + \dots}_{\text{principle part}}$$

case A: all $a_{-n} = 0\ \forall n \Rightarrow z_0$ is called removable

case B: finitely many $a_{-n} \neq 0$: z_0 is called pole of order n, where n is the largest non-zero entry

case C: infinitly many $a_{-n} \neq 0$: z_0 is called essential singularity

12

Panel 13

Classification of Singularities:

13

Panel 14

Ex: a) $f(z) = \frac{z^2 - 2z + 3}{z - 2}$ sing. at $z_0 = 2 \Rightarrow$ pole order 1

b) $g(z) = \frac{1}{z^2(z+1)}$ at $z=0 \Rightarrow$ pole order 2

at $z = -1 \Rightarrow$ pole order 1

c) $h(z) = \frac{\sin(z)}{z^4} = \frac{z - z^3/3! + z^5/5! - \dots}{z^4} \Rightarrow$ pole of order 3

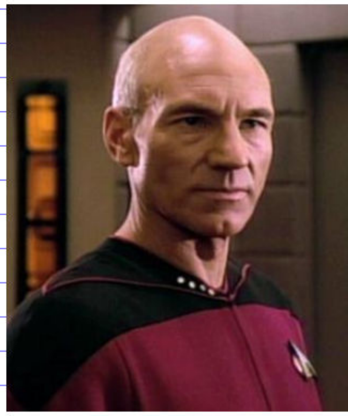
d) $f(z) = \frac{z^2 - 9}{z - 3}$ removable sing. at $z=3$

e) $g(z) = \frac{1 - \cos(z)}{z^2}$ removable singl.!

f) $h(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \Rightarrow$ essential sing.!

14

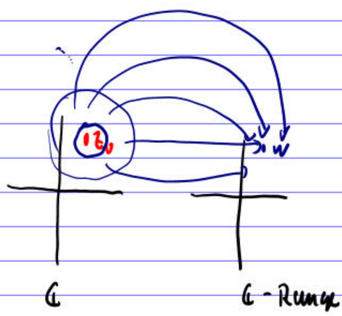
Panel 15



Picard's Theorem:

If f has an essential ← really bad
singularity at z_0 then:

in every nbhd of z_0
f assumes every value
(with one possible exception)
infinitely often.



not for us

15

Panel 16

Finding Residues Thm: f has pole of order m at z_0 iff:

$$f(z) = \frac{g(z)}{(z-z_0)^m}, \text{ with } g \text{ analytic and } g(z_0) \neq 0$$

Ex: $f(z) = \dfrac{5z^2-7z+4}{z^5}$ pole at 0 of order 5

$$= \frac{g(z)}{z^5}, \quad g(z) = 5z^2-7z+4, \ g \text{ analytic}, \ g(0) \neq 0$$

$g(z) = \dfrac{5z^4-3z^3+4z^2}{z^5(z-100)^2}$ at $z=0$: pole of order 3

at $z=100$ pole of order 2

16