Panel 1

(1) Laurent's Theorem

f analytic for $R_1 < |z-z_0| < R_2$. Then

$$f(z) = \dots + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{for } R_1 < |z-z_0| < R_2$$

(2) Diff / Int. Series Thm:

If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ $\forall |z-z_0| < R$:

$$\frac{d}{dz} \sum_{n=0}^{\infty} a_n (z-z_0)^n = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}, \quad |z-z_0| < R$$

$$\int \sum_{n=0}^{\infty} a_n (z-z_0)^n \, dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-z_0)^{n+1}, \quad |z-z_0| < R$$

1

Panel 2

Complex HW

(1) Find the Laurent series centered at $z_0 = 0$, and list specifically the value of the coefficient $a_{-1} =$ ____

a) $f(z) = z^4 \sin(1/z)$ b) $g(z) = \frac{1}{z^4} \sin(z)$

(2) The function $f(z) = \frac{1}{3-z}$ has two series expansions, one for $|z| < 3$ and another for $|z| > 3$. Find them.

(3) $f(z) = \frac{-2}{(z-1)(z-3)}$ has 3 series expansions ($z_0 = 0$). Find all three series and their domain of convergence.

(4) How many series expansions centered at $z_0 = 0$ does $f(z) = \frac{1}{e^z - 1}$ have, and where do they converge?

2

Panel 3

⑤ Use the theorem on differentiation of power series (which says that $\frac{d}{dz}\sum_{n=0}^{\infty} a_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n \frac{d}{dz}(z-z_0)^n = \sum_{n=1}^{\infty} a_n n (z-z_0)^{n-1}$)

to prove that $\frac{d}{dz} \sin(z) = \cos(z)$

⑥ Use differentiation and/or integration to find the series for

$f(z) = \frac{1}{(1-z)^2}$ for $|z| < 1$

3

Panel 4

Complex HW

① Find the Laurent series centered at $z_0 = 0$, and list specifically the value of the coefficient $a_{-1} =$ ___

$$\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} z^{2n+1}$$

a) $f(z) = z^4 \sin(1/z)$

$$= z^4 \left(\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} + \dots \right) = z^3 - \frac{1}{3!}z + \frac{1}{5!}\frac{1}{z} - \dots$$

$$= z^4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-2n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-2n+3}, \quad a_{-1} = \frac{1}{5!}$$

b) $g(z) = \frac{1}{z^4} \sin(z)$

$$= \frac{1}{z^4}\left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \dots \right) = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}z - \dots$$

$$= \frac{1}{z^4} \sum_{0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-3}, \quad a_{-1} = -\frac{1}{3!}$$

4

Panel 5

② The function $f(z) = \frac{1}{3-z}$ has two series expansions, one for $|z| < 3$ and another for $|z| > 3$. Find them.

A) $\frac{1}{3-z} = \frac{1}{3(1-z/3)} = \frac{1}{3}\sum_{n=0}^{\infty} (z/3)^n = \underline{\underline{\sum_{n=0}^{\infty} z^n/3^{n+1}}}, \ \underline{|z|<3}$

B) $\frac{1}{3-z} = \frac{1}{-z(1-3/z)} = \frac{1}{-z}\sum_{n=0}^{\infty} \left(\frac{3}{z}\right)^n = \underline{\underline{-\sum_{n=0}^{\infty} 3^n/z^{n+1}}}, \ \underline{|z|>3}$

5

Panel 6

③ $f(z) = \frac{-2}{(z-1)(z-3)}$ has 3 series expansions. Find all three series and their domain of convergence.

First note: $\frac{-2}{(z-1)(z-3)} = \frac{1}{z-1} - \frac{1}{z-3}$

Ⓐ $\frac{1}{z-1} = -\sum_{n=0}^{\infty} z^n$ and $\frac{1}{z-3} = -\frac{1}{3(1-z/3)} = -\sum_{n=0}^{\infty} z^n/3^{n+1}$

$\Rightarrow \frac{-2}{(z-1)(z-3)} = \sum_{n=0}^{\infty} \left(\frac{1}{3^{n+1}} - 1\right) z^n \quad \forall\ |z| < 1$

Ⓑ $\frac{1}{z-1} = \frac{1}{z(1-1/z)} = \sum_{n=0}^{\infty} (1/z)^{n+1}$ and $\frac{1}{z-3} = -\sum_{n=0}^{\infty} z^n/3^{n+1}$

$\Rightarrow \frac{-2}{(z-1)(z-3)} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^{n+1} + \sum_{n=0}^{\infty} z^n/3^{n+1} \quad \forall\ 1 < |z| < 3$

6

Panel 7

$$\frac{-2}{(z-1)(z-3)} = \frac{1}{z-1} - \frac{1}{z-3}$$

(c) $\frac{1}{z-1} = \frac{1}{z(1-1/z)} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^{n+1}$

$\frac{1}{z-3} = \frac{1}{z(1-3/z)} = \sum_{n=0}^{\infty} 3^n / z^{n+1}$,

$\Rightarrow \frac{-2}{(z-1)(z-3)} = \underline{\sum_{n=0}^{\infty} (1-3^n) \, 1/z^{n+1}} \quad \forall \, |z|>3$

7

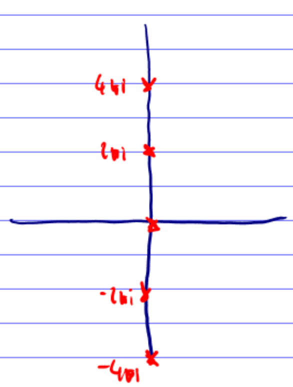
Panel 8

(4) How many series expansions centered at $z_0=0$ does $f(z) = \frac{1}{e^z-1}$ have, and where do they converge?

$f(z) = 1/(e^z-1)$ is <u>not</u> diff'ble when

$e^z - 1 = 0$ or $e^z = 1$ or $z = 0, 2\pi i, 4\pi i, \dots$

$\Rightarrow$ inf. many series, converging



$0 < |z| < 2\pi$ or

$2\pi < |z| < 4\pi$ or

$4\pi < |z| < 6\pi$ ...

8

Panel 9

⑤ Use the theorem on differentiation of power series (which says that $\frac{d}{dz}\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n \frac{d}{dz}(z-z_0)^n = \sum_{n=1}^{\infty} a_n n (z-z_0)^{n-1}$)

to prove that $\frac{d}{dz}\sin(z) = \cos(z)$

$$\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} z^{2n+1}$$

$$\Rightarrow \frac{d}{dz}\sin(z) = 1 - \frac{1}{3!}3z^2 + \frac{1}{5!}5z^4 - \dots = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\cdot(2n+1)z^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

$$= \cos(z)$$

9

Panel 10

ⓒ Use differentiation and/or integration to find the series for

$$f(z) = \frac{1}{(1-z)^2} \quad \text{for } |z| < 1$$

Know $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots$

$$\Rightarrow \frac{d}{dz}(1-z)^{-1} = +(1-z)^{-2} = \frac{d}{dz}\sum_{n=0}^{\infty} z^n$$

$$\Rightarrow \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} = 1 + 2z + 3z^2 + 4z^3 + \dots$$

$$\forall |z| < 1$$

10

Panel 11

f analytic for $\underline{0} < |z - z_0| < R$. Then

$$f(z) = \ldots + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

$$= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{for } 0 < |z-z_0| < R$$

This is only half of Laurent's theorem. The missing part says: that

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad n = 0, \pm 1, \pm 2, \ldots$$

Thus, for example:

$$a_2 = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz$$

and

$$a_{-1} = \frac{1}{2\pi i} \int_C f(z)\, dz \; !$$

interesting! The a_{-1} coeff. is equal to the integral!!!

11

Panel 12

Ex: Use the fact that $a_{-1} = \frac{1}{2\pi i} \int_C f(z)\,dz$ to find

a) $\int_{|z|=1} \sin(z)\,dz$ b) $\int_{|z|=1} \frac{1}{z^4}\sin(z)\,dz$ c) $\int_{|z|=1} z^4 \sin(1/z)\,dz$

a) $\sin(z)$ is analytic and has a Taylor series $\Rightarrow a_{-1} = 0$

$$\Rightarrow \int_{|z|=1} \sin(z)\,dz = 0$$ (also follows from Cauchy Goursat)

b) $\int_{|z|=1} \frac{1}{z^4}\sin(z)\,dz = 2\pi i\, a_{-1} = 2\pi i \left(-\frac{1}{3!}\right)$ from before.

Also: $\int_{|z|=1} \frac{\sin(z)}{z^4}\,dz = \frac{2\pi i}{3!} f'''(0) = \frac{2\pi i}{3!}(-1)$ by General Cauchy Int.

12

Panel 13

So a) and b) could be handled by other theorems. But

c) $\int_{|z|=1} z^4 \sin\left(\frac{1}{z}\right) dz$ has no theorem that applies.

But since $a_{-1} = 1/5!$ we have

$$\int_{|z|=1} z^4 \sin\left(\frac{1}{z}\right) dz = 2\pi i \; 1/5!$$

So the a_{-1} coefficient in the Laurent series is important and gets a special name:

13

Panel 14

Def: If f is analytic in $0 < |z-z_0| < R$ and $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ for $0 < |z-z_0| < R$, we define the Residue of f at z_0, or

$$\mathrm{Res}(f, z_0) = a_{-1}$$

Note: $\mathrm{Res}(f, z_0)$ requires that f is analytic in $0 < |z-z_0| < R$ and must be written in powers of $(z-z_0)^n$ to read off the residue.

14

Panel 15

Ex: Consider $f(z) = \frac{2}{z(z-1)}$. Let's say we want to find a) Res$(f, 0)$ b) Res$(f, 1)$ and c) Res$(f, 2)$

a) To find Res$(f, 0)$ we need to write $\frac{2}{z(z-1)}$ in terms of powers of z^n. Thus

$$\frac{2}{z(z-1)} = -\frac{2}{z} \cdot \frac{1}{1-z} = -\frac{2}{z} \sum_{n=0}^{\infty} z^n = -2 \sum_{n=0}^{\infty} z^{n-1}$$

Thus, Res$(f, 0) = \underline{-2}$

15

Panel 16

b) To find Res$(f, 1)$ for $f(z) = \frac{2}{z(z-1)}$ we need powers of $(z-1)^n$! Thus:

$$\frac{2}{z(z-1)} = \frac{2}{z-1} \cdot \frac{1}{z} = \frac{2}{z-1} \frac{1}{1+z-1} =$$

$$= \frac{2}{(z-1)} \sum_{n=0}^{\infty} (-1)^n (z-1)^n =$$

$$= 2 \sum_{n=0}^{\infty} (-1)^n (z-1)^{n-1}$$

Thus: Res$(f, 1) = 2$

c) $f(z) = \frac{2}{z(z-1)}$ is analytic near 2. Thus

$\frac{2}{z(z-1)} = \sum_{n=0}^{\infty} a_n (z-2)^n$. Thus Res$(f, 2) = 0$

16

Panel 17

We have worked out $Res(f,0) = -2$, $Res(f,1) = 2$ for $f(z) = \frac{2}{z(z-1)}$. Use these residues to find:

$$\int_C \frac{2}{z(z-1)}\,dz \quad \text{if}$$

a) C is circle center $z_0 = 0$, radius $1/2$

b) C is circle center $z_0 = 1$, radius $1/2$

c) C is circle center $z_0 = 2$, radius $1/2$

(relatively) easy

d) C is circle center $z_0 = i$, radius 10

Try your best guess

17

Panel 18

HW Find the residues for the functions as given:

a) $Res(f, 0)$, $f(z) = z^5 \cos(1/z)$

b) $Res(f, 0)$, $f(z) = z^2 e^{1/z}$

c) $Res(f, 0)$, $f(z) = \frac{3}{z(z+2)}$

d) $Res(f, -2)$, $f(z) = \frac{3}{z(z+2)}$

e) $Res(f, 3)$, $f(z) = \frac{3}{z(z+2)}$

f) $Res(f, 0)$, $f(z) = \frac{1}{z^3(z-2)}$

18