Panel 1

Taylor's Theorem: f analytic in $|z-z_0|<R$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \forall\ |z-z_0|<R$$

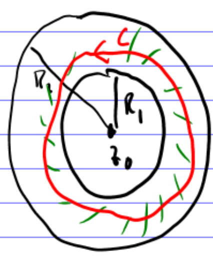
where $a_n = f^{(n)}(z_0)/n!$ useful

Laurent's Theorem: f analytic in $R_1 < |z-z_0| < R_2$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad R_1 < |z-z_0| < R_2$$

where $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

useful NOT



$z_0 = 0$:

$n=2$: $a_2 = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^3}\,dz = \frac{f^{(2)}(0)}{2!}$

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Panel 2

Note: If f is analytic inside $|z-z_0|<R_2$, and this inside Curve C then

$$a_n = \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{f^{(n)}(z_0)}{n!}, \quad n \geq 0$$

and $a_n = 0 \quad \forall\ n<0$ by Cauchy Goursat!

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Panel 3

Special ~~Power~~ Taylor Series everyone should know:

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots$$

$$f(z) = e^z = \sum_{n=0}^{\infty} z^n/n! = 1 + z + z^2/2! + z^3/3! + \dots$$

$$f(z) = \cos(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n}/(2n)! = 1 - z^2/2! + z^4/4! - z^6/6! + \dots$$

$$f(z) = \sin(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n+1}/(2n+1)! = z - z^3/3! + z^5/5! - z^7/7! + \dots$$

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Panel 4

Complex HW

① Find the McLaurin series (i.e. the Taylor series centered at $z_0 = 0$) for the following functions:

a) $f(z) = \cosh(z)$ b) $g(z) = \frac{z}{9+z^4}$

c) $h(z) = z^3 e^{2z}$

② Find the Taylor series for $f(z) = \frac{1}{1-z}$ centered at $z_0 = i$. What is the radius of convergence?

③ Find $\lim_{z \to 0} \frac{z^2 \sin(z) - z^3}{z^5}$ without l'Hospital's Rule

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Panel 5

(4) Find the radius of convergence of the Taylor series centered at z_0 for:

a) $f(z) = \frac{1}{2-z}$ for $|z+1| < R$ (i.e. $z_0 = -1$)

b) $f(z) = \frac{z}{(z+1)(z-2)}$ for $|z-(1+i)| < R$ (i.e. $z_0 = 1+i$)

c) $f(z) = \frac{1}{e^z - i}$ for $|z| < R$ (i.e. $z_0 = 0$)

<u>Note</u>: You do <u>not</u> have to actually find the series, you can find R by drawing appropriate pictures

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Panel 6

Some special Laurent Series centered at zero

$$f(z) = e^{1/z} = 1 + \left(\frac{1}{z}\right) + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!}$$

$$g(z) = z^3 \cos\left(\frac{1}{3z}\right)$$

$$\cos(z) = 1 - z^2/2! + z^4/4! - z^6/6! + \dots$$

$$\cos(1/3z) = 1 - \frac{z^{-2}}{3^2 2!} + \frac{z^{-4}}{3^4 4!} - \dots$$

$$h(z) = \frac{\cos(z)}{\sin(z)}$$

$$z^3 \cos(1/3z) = z^3 - \frac{z}{3^2 2!} + \frac{z^{-1}}{3^4 4!} + \dots$$

no z^{-2}

$$\frac{1}{2\pi i} \int z \cdot \left(z^3 \cos(1/3z)\right) dz = \boxed{a_{-2}} = 0$$

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Panel 7

$$e^{z} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots = \sum_{n=0}^{\infty} z^{-n}/n! = \sum_{n=-\infty}^{0} z^{n}/(-n)!$$

$$\frac{1}{2\pi i}\int_{|z|=1} z e^{1/z}\, dz = a_{-2} = 1/2$$

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\, dz \quad n = -2$$

a_{-2} goes with z^{-2}

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Panel 8

$$f(z) = \frac{\cos(z)}{\sin(z)} = \sum_{n=-\infty}^{\infty} a_n z^n \qquad 0 < |z| < \pi$$

$$= \sum_{n=-\infty}^{\infty} b_n z^n \qquad \pi < |z| < 2\pi$$

cos(z) = $1/z$ ← cos

sin $\quad z - z^3/3! + z^5/5! - z^7/7! \,\big|\, 1 - z^2/2! + z^4/4! - z^6/6! + \dots$

$1 - z^2/3! + z^4/5! \dots$

Going nuts! (later)

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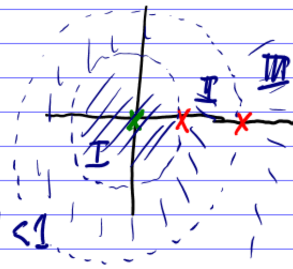
Panel 9

Consider $f(z) = \dfrac{-1}{(z-1)(z-2)}$



Write
$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad |z| < 1$$
$$= \sum_{n=-\infty}^{\infty} b_n z^n, \quad 1 < |z| < 2$$
$$= \sum_{n=-\infty}^{\infty} c_n z^n, \quad 2 < |z|$$

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Panel 10

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$
$$= -\frac{1}{1-z} + \frac{1}{2-z}$$
$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2(1-z/2)} =$$
$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} z^n/2^{n+1} =$$
$$\sum_{n=0}^{\infty} \left(-1 + \frac{1}{2^{n+1}}\right) z^n, \quad |z| < 1$$

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Panel 11

11

Panel 12

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}, \quad 1<|z|<2$$

$$= \frac{1}{z(1-1/z)} + \frac{1}{2(1-z/2)} =$$

$\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \dots$

$z^{-1} + z^{-2} + \dots$

$\sum_{n=-\infty}^{-1} z^n$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} z^n/2^{n+1} =$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1} + \sum_{n=0}^{\infty} z^n/2^{n+1} =$$

$$= \sum_{n=-\infty}^{-1} z^n + \sum_{n=0}^{\infty} z^n/2^{n+1} \quad \#$$

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Panel 13

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2} \quad , \ 2 < |z|$$

$$= \frac{1}{z(1-1/z)} - \frac{1}{z(1-2/z)} =$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n - \frac{1}{z} \sum_{n=0}^{\infty} \left(2/z\right)^n =$$

$$= \sum_{n=0}^{\infty} 1/z^{n+1} - \sum_{n=0}^{\infty} 2^n/z^{n+1} =$$

$$= \sum_{n=0}^{\infty} \left(1 - 2^n\right) 1/z^{n+1}$$

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Panel 14

$f(z) = \frac{1}{3-z}$ has two series centered at 0

$$\frac{1}{3(1-z/3)} = \text{Taylor} \quad , \ |z| < 3$$

$$\frac{1}{3-z}$$

$$\frac{1}{z(3/z-1)} = \text{Laurent} \quad , \ |z| > 3$$

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Panel 15

How many Laurent series centered at $z_0 = 0$ for

$$f(z) = \frac{1}{z(2-z)}$$

2 series, both Laurent!

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Panel 16

Thm: If $\sum_{n=0}^{\infty} a_n (z-z_0)$ converges for $|z-z_0| < R$

then the limit function is analytic inside the circle of convergence and:

$$\frac{d}{dz} \sum_{n=0}^{\infty} a_n (z-z_0)^n = \sum_{n=0}^{\infty} \frac{d}{dz} a_n (z-z_0)^n =$$

$$= \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

$$\int \sum_{n=0}^{\infty} a_n (z-z_0)^n \, dz = \sum_{n=0}^{\infty} \int a_n (z-z_0)^n \, dz =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} a_n (z-z_0)^{n+1}$$

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Panel 17

Find $\frac{d}{dz} e^z = \frac{d}{dz}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots\right)$

$$= 0 + 1 + 2\frac{z}{2!} + 3\frac{z^2}{3!} + 4\frac{z^3}{4!} + \dots$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = e^z$$

$$= \frac{d}{dz}\sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} n\frac{z^{n-1}}{n!} =$$

$$= \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$= e^z$$

more later

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