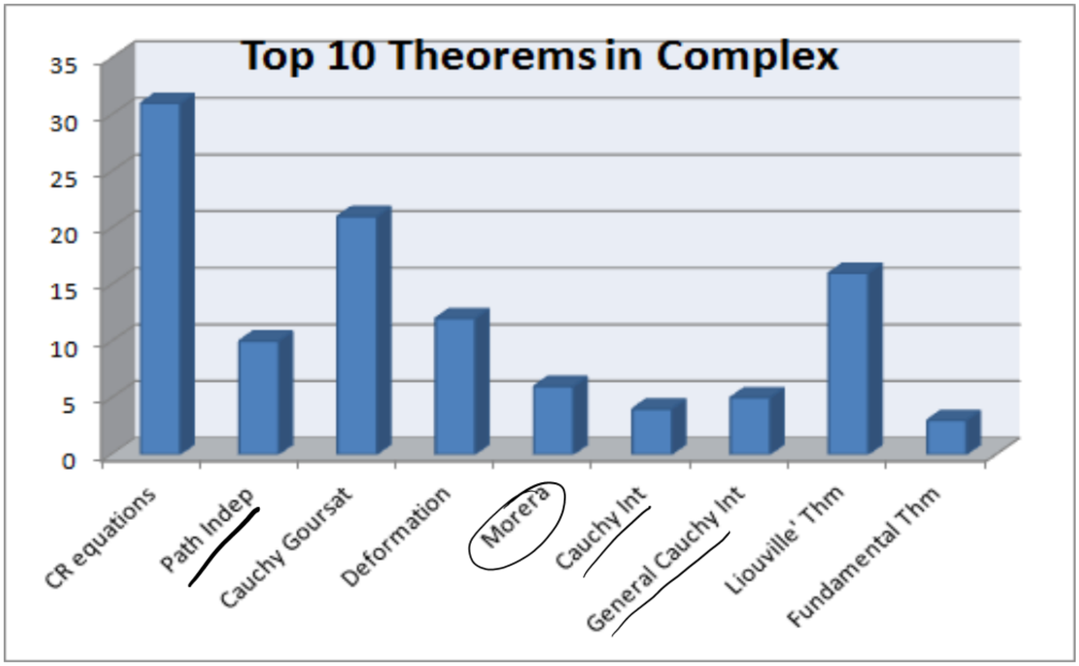
Panel 1



Last: Cauchy Inequality

1

Panel 2

$$\int_C f(z)\,dz = F(b) - F(a)$$

$\int_1^i \bar{z}\,dz$ depends on the path you pick from

1 to i!

2

Panel 3

Last Time:

Power Series. $\sum a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$

Taylor's thm: If f is analytic in $|z-z_0|<R$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \forall\ |z-z_0|<R$$

memorize this

and $a_n = \frac{f^{(n)}(z_0)}{n!}$ (useful in emergencies) $\qquad \frac{1}{1-z/w}$

Proof. $f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_C \frac{f(w)}{w(1-\frac{z}{w})}\,dw = \frac{1}{2\pi i}\int \frac{f(w)}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n dw$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(w)}{w^{n+1}}\,dw\right) z^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!} z^n \qquad (z_0 = 0)$$

3

Panel 4

Special Power Series every educated ~~person~~ mathematician must know:

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots$$

$$f(z) = e^z = \sum_{n=0}^{\infty} z^n/n! = 1 + z + z^2/2! + z^3/3! + \dots$$

$$f(z) = \cos(z) = \sum_{n=0}^{\infty} (-1)^n\, z^{2n}/(2n)! = 1 - z^2/2! + z^4/4! - z^6/6! + \dots$$

$$f(z) = \sin(z) = \sum_{n=0}^{\infty} (-1)^n\, z^{2n+1}/(2n+1)! = z - z^3/3! + z^5/5! - z^7/7! + \dots$$

4

Panel 5

Find McLaurin Series for $f(z) = 1/1-z$, $g(z) = z/z^2+4$

$\shortparallel$

Taylor series with $z_0 = 0$

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad , |z|<1$$

$$g(z) = \frac{z}{z^2+4} = z \frac{1}{4-(-z^2)} = \frac{z}{4} \frac{1}{1-\left(-\frac{z^2}{4}\right)} = \frac{z}{4} \sum \left(-\frac{z^2}{4}\right)^n =$$

$$= \frac{z}{4} \sum (-1)^n \frac{z^{2n}}{4^n} =$$

$$= \sum_{n=0}^{\infty} (-1) \frac{z^{2n+1}}{4^{n+1}} \quad \text{for } \left|-\frac{z^2}{4}\right|<1$$

$|z|^2<4$

$|z|<2$

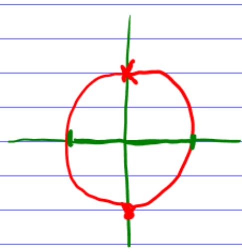
5

Panel 6

$$\text{In } \mathbb{R}: \quad f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1) x^{2n} \; , |x|<1$$

$$\frac{1}{1-(-x^2)}$$

$|z|<1$



6

Panel 7

Find McLaurin Series for $f(z) = z e^{z^2}$

$$e^z = \sum_{n=0}^{\infty} z^n/n! \quad \Rightarrow \quad e^{z^2} = \sum_{n=0}^{\infty} z^{2n}/n!$$

$$\Rightarrow f(z) = z e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{n!}$$

Prove that.

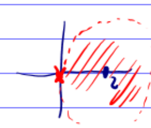
$$\lim_{z \to 0} \frac{\sin(z)}{z} = \lim \frac{1}{z} \sin(z) = \lim_{z \to 0} \frac{1}{z}\left(z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \dots\right) = \lim_{z \to 0}\left(1 - \cancel{\tfrac{1}{3!} z^2}^{\,0} + \cancel{\tfrac{1}{5!} z^4}^{\,0} - \cancel{\dots}\right) = 1$$

$$\lim_{z \to 0} \frac{\cos(z) - 1}{z} = \lim \frac{1}{z}\left(\cancel{1} - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots - \cancel{1}\right) = \lim_{z \to 0}\left(-\frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{5!} \dots\right) = 0$$

Panel 8

Find power series for $f(z) = 1/z$ centered at $c = \cancel{0}\ 2$

$$f(z) = \frac{1}{z} = \sum_{n=0}^{\infty} a_n (z-2)^n, \quad |z-2| < 2$$



$$\frac{1}{z} = \frac{1}{2 + z - 2} = \frac{1}{2} \frac{1}{1 + \frac{z-2}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{z-2}{2}\right)^n = \qquad \left|\frac{z-2}{2}\right| < 1$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^n}{2^n} =$$

$$= \sum_{n=0}^{\infty} (-1) \frac{(z-2)^n}{2^{n+1}}, \quad |z-2| < 2$$

Panel 9

Find a series expansion for

$$f(z) = \frac{1}{z^3 - z^4} = \frac{1}{z^3(1-z)} = \frac{1}{z^3} \sum_{n=0}^{\infty} z^n \qquad 0 < |z| < 1$$

$$= 1/z^3 \, (1 + z + z^2 + z^3 + \dots) =$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \dots \qquad 0 < |z| < 1$$

9

Panel 10

Radius of Convergence Theorem:

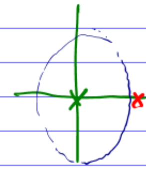
① If $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a power series, then it converges for $|z - z_0| < R$ where $R = \lim \left| \frac{a_n}{a_{n+1}} \right|$ ~~(take 2)~~

② If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a Taylor series, then it converges for $|z - z_0| < R$ where R is largest radius where f is analytic!

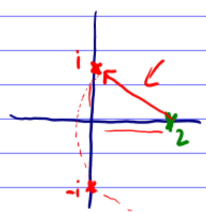
10

Panel 11

Ex: If $\frac{1}{2-z} = \sum_{n=0}^{\infty} a_n z^n$ for $|z|<R$, find $R=2$



Ex: If $\frac{1}{1+z^2} = \sum_{n=0}^{\infty} a_n (z-2)^n$ for $|z-2|<R$, find R



$R = \sqrt{5}$

11

Panel 12

If f is analytic at $z_0 \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $|z-z_0|<R$

Taylor's Thm

positive powers

If f is analytic in $\overset{R_1}{0} < |z-z_0| < \overset{R_2}{R}$ will have

series with pos. + negative powers!

Laurent series

12

Panel 13

Laurent Theorem: If f is analytic in a ring-shaped domain $R_1 < |z - z_0| < R_2$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \quad R_1 < |z-z_0| < R_2$$

with

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad \left(= \frac{f^{(n)}(z_0)}{n!} \right)$$

and

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}}\, dz, \quad n = 1, 2, 3, \ldots$$

13

Panel 14

In short:

f analytic on $|z - z_0| < R$ then: $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$ semi-useful

f analytic in $\overbrace{R_1 < |z-z_0| < R_2}^{\text{washer}}$, then:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \ldots + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \ldots$$

not useful

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

14

Panel 15

Ex: Find Laurent series for

$f(z) = e^{1/z}$ centered at $z_0 = 0$

$e^z = \sum_{n=0}^{\infty} z^n/n! = 1 + z + z^2/2! + z^3/3! + \dots$

$\Rightarrow e^{1/z} = \sum_{n=0}^{\infty} (1/z)^n/n! = 1 + 1/z + 1/(z^2 \cdot 2!) + 1/(z^3 3!) + \dots = \sum_{n=-\infty}^{0} \frac{z^n}{n!}$

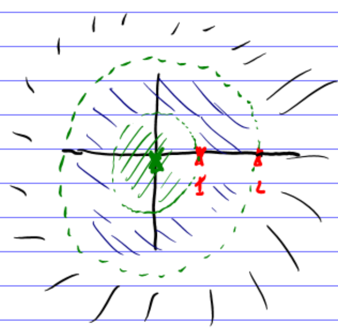
$g(z) = \frac{1}{(z-i)^2}$ centered at $z_0 = i$

$\frac{1}{(z-i)^2} = \sum_{n=-\infty}^{\infty} c_n (z-i)^n \Rightarrow$ Thus: $c_{-2} = 1$, all other $c_n = 0$

15

Panel 16

Consider $f(z) = \frac{-1}{(z-1)(z-2)}$



Write $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$

1. Series converges for $|z| < 1$ $\Rightarrow$ Taylor series (only positive powers)
2. Series converges for $1 < |z| < 2$ $\Rightarrow$ Laurent series with pos/neg powers
3. Series converges for $|z| > 2$ $\Rightarrow$ Laurent series with neg powers!

Find them: next time!!

16