Panel 1

Foundational Theorems of Complex Analysis

① Cauchy-Riemann Equations

* ② Cauchy-Goursat Thm — path closed or not?

* ③ Deformation thm — If not → ④

* ④ Path Independence thm — If closed: ②

* ⑤ Cauchy Integral Formula — ⑤ $\int \frac{?}{z-\#}$

* ⑥ General Cauchy Int. Formula — ⑥ $\int \frac{?}{(z-\#)^?}$

⑦ Morera's thm

⑧ Cauchy's Inequality — ⑧ or "by hand"

⑨ Liouville's thm

⑩ Fund. thm. of Algebra

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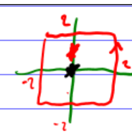
Panel 2

Complex Homework



① Let C be the boundary of the square with sides at $x = \pm 2$ and $y = \pm 2$. Compute.

a) $\int_C \frac{e^{-z}}{z - \frac{\pi i}{2}}\,dz = 2\pi i\, f\left(\frac{\pi i}{2}\right) = 2\pi i\, e^{-\pi i/2} = 2\pi i(-i) = \underline{\underline{2\pi}}$

b) $\int \frac{z}{2z+1}\,dz = \int_C \frac{z}{2(z+\frac{1}{2})}\,dz = \frac{1}{2}\int_C \frac{z}{z+\frac{1}{2}}\,dz = \frac{1}{2}\cdot 2\pi i\, f\left(-\frac{1}{2}\right) = -\frac{\pi i}{2}$

$\int \frac{\cosh(z)}{z^4}\,dz = \int \frac{f(z)}{(z-0)^4}\,dz = \frac{2\pi i}{3!} f'''(z)\Big|_{z=0} = 0$

$\int \frac{2}{z^2-1}\,dz = \int \frac{1}{z-1} - \frac{1}{z+1}\,dz = 2\pi i - 2\pi i\, 1 = 0$

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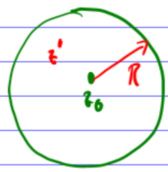
Panel 3

(3) Prove that if f is entire and $|f(z)| \le M|z|$ for all $z \in \mathbb{C}$ then $f(z) = cz$, c some constant.

Want: $|f^{(n)}(z_0)| \le \frac{n!M}{R^n}$ on $|z - z_0| < R$

Pick z_0. Pick any R. Know $|f(z)| \le M|z|$ $\forall z$ (?)

on this disk $|f(z)| \le M(|z_0| + R)$ $\forall z \in C_R(z_0)$



$$\Rightarrow |f'(z_0)| \le \frac{M(|z_0|+R)}{R} \xrightarrow[R\to\infty]{} M$$

$$|f''(z_0)| \le \frac{2M(|z_0|+R)}{R^2} \xrightarrow[R\to\infty]{} 0$$

$\Rightarrow f''(z) = 0$ $\forall z$ $\Rightarrow f'(z) = c$ $\Rightarrow f(z) = cz +$ But $f(0) = 0$ q.e.d.

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Panel 4

There is one more important Theorem, called the **Maximum Modulus Principle**.

<u>Recall:</u> If f is a function of 2 real variables

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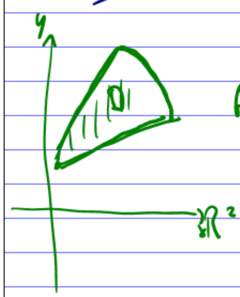
Panel 5

Maximum Modulus Principle: If f is analytic in a domain D and not constant, then $|f(z)|$ has no max. inside D

complex

Recall: $f(x,y)$, $(x,y) \in \mathbb{R}^2$ in a domain D, D closed + bdd, then f has abs. max and min.

Real



$f(x,y)$

① $\nabla f = (f_x, f_y) = (0,0)$

② Check bdry

Max. could occur inside D at a critical point or on the bdry of D

$\mathbb{R}$: $f: [a,b] \to \mathbb{R}$ has max/min.

$f' = 0$

bdry points

$f(x) = x^3 - 2x$ on $[0,1]$

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Panel 6

Ex: Find abs. max. of $|f(z)|$, where $f(z) = z^2$, over the unit disk. $z = x + iy$

$|z^2| = |z|^2$

$= \left(\sqrt{x^2+y^2}\right)^2 =$

$= x^2 + y^2$

$z^2 = \underline{x^2 - y^2} + \underline{2ixy}$

$|z^2| = \sqrt{(x^2-y^2)^2 + (2xy)^2} =$

$= \sqrt{x^4 - 2x^2y^2 + y^4 + 4x^2y^2} =$

$= \sqrt{x^4 + 2x^2y^2 + y^4} = \sqrt{(x^2+y^2)^2} = x^2 + y^2$

① Critical: $f_x = 2x = 0 \Rightarrow x = 0$

$f_y = 2y = 0 \Rightarrow y = 0$

critical point $(0,0)$

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Panel 7

$f(x,y) = x^2+y^2$ $\qquad$ $f_x = 2x$, $f_{xx} = 2$

critical is $(0,0)$

$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \Rightarrow D = f_{xx}\cdot f_{yy} - f_{xy}^2$, i.e.

Hessian matrix

$H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $D = 4$ $\qquad$ if $D>0$ and $f_{xx}>0 \Rightarrow$ min

on ∂D : $x^2+y^2=1$ so max of $f(x,y) = x^2+y^2$ is 1 on the disk!

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Panel 8

Corollary: If f is analytic on D and continuous on closure of D, and D is bounded, then f has a max. on the boundary of D!

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Panel 9

Complex Analysis

Algebra in $\mathbb{C}$ — Highlight: Euler's Equation $e^{\pi i} + 1 = 0$

Derivatives in $\mathbb{C}$ — CR equations

Integrals in $\mathbb{C}$ — Cauchy - Int. Formula

i — be rational

π — get real

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Panel 10

Infinite Series: The formal sum

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \dots + z_n + \dots$$

$$\sum 1 = 1+1+1+\dots+1+\dots = \infty$$

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots = \infty$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \underbrace{\underbrace{\underbrace{1 + \frac{1}{2}}_{1.5} + \frac{1}{4}}_{1.75} + \frac{1}{8}} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots = 2$$

strange

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Panel 11

Def: The formal sum

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + \dots$$

is called infinity series. The series is convergent if the sequence of partial sums:

$$S_N = z_0 + z_1 + z_2 + \dots + z_N$$

converges as a sequence.

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Panel 12

Ex: Find out if the sum $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ converges.

$S_0 = 1$

YES, TO 2

$S_1 = 1 + 1/2$

$S_2 = 1 + 1/2 + 1/4$

$S_3 = 1 + 1/2 + 1/4 + 1/8$

$\vdots$

$S_N = 1 + \cancel{1/2} + \cancel{1/4} + \cancel{\dots} + \cancel{1/2^{N-1}} + \cancel{1/2^N}$

$- \Big(\frac{1}{2} S_N = \cancel{1/2} + \cancel{1/4} + \cancel{1/8} + \dots + \cancel{1/2^N} + 1/2^{N+1}$

$S_N(1 - 1/2) = S_N - \frac{1}{2} S_N = 1 - 1/2^{N+1}$

$$S_N = \frac{1 - 1/2^{N+1}}{1 - 1/2} \xrightarrow[N \to \infty]{} \frac{1}{1 - 1/2} = \frac{1}{1/2} = \underline{\underline{2}}$$

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