Panel 1

Foundational Theorems of Complex Analysis

1. Cauchy-Riemann Equations
2. Cauchy-Goursat Thm — (3 out of 10 - quiz next time)
3. Deformation thm
4. Path Independence thm
5. Cauchy Integral Formula
6. General Cauchy Int. Formula
7. Morera's thm
8. Cauchy's Inequality
9. Liouville's thm
10. Fund. thm. of Algebra

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Panel 2

Cauchy's Inequality: f analytic in a simply connected domain D, $z_0 \in D$, $C_R(z_0) \subset D$, and $|f(z)| \le M$ on $C_R(z_0)$.

Then
$$|f^{(n)}(z_0)| \le \frac{Mn!}{R^n}$$

Proof. Know:
$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$\Rightarrow |f^{(n)}(z_0)| \le \frac{n!}{2\pi} \cdot \frac{M}{R^{n+1}} \cdot 2\pi R = \frac{Mn!}{R^n}$$

($2\pi R$ = length of $C_R(z_0)$)

$$\left|\frac{f(z)}{(z-z_0)^{n+1}}\right| = \frac{|f(z)|}{|z-z_0|^{n+1}} \le \frac{M}{R^{n+1}} \qquad z \in C_R(z_0), \text{ i.e. } |z-z_0| = R$$

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Panel 3

Liouville's Theorem: If f is entire and bounded, then f must be constant.

Pick any $z_0 \in \mathbb{C}$, any R. Then f is analytic on $C_R(z_0)$

$\Rightarrow$ by Cauchy's Inequality:

$$|f'(z_0)| \leq \frac{M}{R} \qquad \forall R.$$

So make $R \to \infty \Rightarrow |f'(z_0)| = 0$

$$\Rightarrow f'(z_0) = 0 \qquad \forall z_0$$

$$\Rightarrow f'(z) \equiv 0 \qquad \Rightarrow f \text{ is const.}$$

q.e.d.

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Panel 4

Fundamental Theorem of Algebra: If $p(z)$ is a polynomial of degree $n \geq 1$, then $p(z) = 0$ for at least one $z_0 \in \mathbb{C}$.

Proof: $p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$

Say $p_n(z) \neq 0$ for any z

Then $f(z) = 1/p(z)$ is entire

$$\frac{1}{p(z)} = \frac{1}{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0} = \frac{1}{z^n \left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)}$$

Note: $\lim_{|z| \to \infty} a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} = a_n$

$\Rightarrow \exists R$ s.t. $\left|a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n}\right| \geq |a_n| - 1 \qquad \forall |z| \geq R$

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Panel 5

But then $|f(z)| = \left|\frac{1}{z^n(a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^{n}})}\right| \leq \frac{1}{|z|^n(|a_n|+1)} = \frac{1}{R^n(|a_n|-1)}$

$\forall |z| \geq R$

But f is cont. $\Rightarrow \exists M$ s.t. $|f(z)| \leq M$ on $|z| \leq R$

Pick $K = \max\{M, \frac{1}{R^n(|a_n|-1)}\}$. Then

$|f(z)| \leq K \quad \forall z$

$\Rightarrow f(z) = \text{const.}$

$\Rightarrow 1/p(z) = c \Leftrightarrow p(z) = 1/c$, i.e. degree $n = 0$ ↯ to $p(z) \neq 0$

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Panel 6

Consequence of Fund. Thm. of Algebra: If $p_n(z)$ is a complex polynomial of degree n it has exactly n zero (counting multiplicity).

Why: $p(z)$ any polyn. of degree n

$\Rightarrow p(z) = 0$ for z_0

$\Rightarrow p(z) = (z-z_1)\, q(z)$, q is polyn. of degree n-1

$\Rightarrow p(z) = (z-z_1)(z-z_2)\tilde{q}(z)$, $\tilde{q}$ is a polyn. of degree n-2.

⋮

$p(z) = (z-z_1)(z-z_2)\dots(z-z_n)\,c$

$p(z) = 2z^2 - 4z + 8 = 0$

$z^n + 1 = 0 \Rightarrow$ n-th roots of unity

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Panel 7

More Brain Gymnastics:

① If f is entire and $|f(z)| \le M|z|$ then $f(z) = cz$

② Let $g(z) = \int_{|s|=3} \frac{2s^2 - s - 2}{s - z}\,ds$, $|z| \ne 3$

Find $g(2)$ and $g(5)$

③ Let C be the circle $|z-i| = 2$. Find

a) $\int_C \frac{1}{z^2+4}\,dz$ b) $\int_C \frac{1}{(z^2+4)^2}\,dz$

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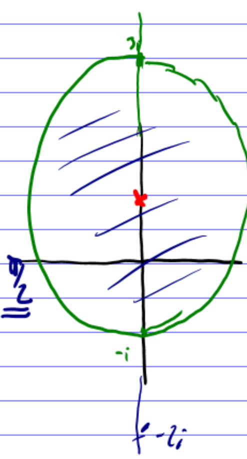
Panel 8

③ Let C be the circle $|z-i| = 2$. Find

$\int_C \frac{1}{z^2+4}\,dz =$

$= \int_C \frac{1}{(z+2i)(z-2i)}\,dz = \int \frac{f(z)}{z-2i}\,dz =$ (I: $\frac{1}{z+2i} = f(z)$, II: $z-2i$)

$= 2\pi i\, f(2i) = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}$



b) $\int_C \frac{1}{(z^2+4)^2}\,dz = \int \frac{1}{(z+2i)^2(z-2i)^2}\,dz$ ($f(z) = \frac{1}{(z+2i)^2}$)

$= \int \frac{f(z)}{(z-2i)^2}\,dz = \frac{2\pi i}{1!} f'(2i) = 2\pi i \cdot \frac{1}{32i} = \frac{\pi}{16}$

$f(z) = (z+2i)^{-2}$, $f'(z) = -2(z+2i)^{-3}\Big|_{z=2i} = -2(4i)^{-3} = +\frac{2}{i\,64} = \frac{1}{32i}$

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Panel 9

② Let $g(z) = \int_{|s|=3} \frac{2s^2 - s - 2}{s - z}\, ds$, $|z| \neq 3$

$g(5) = \int_{|s|=3} \frac{2s^2-s-2}{s-5}\, ds = 0$

$g(2) = \int_{|s|=3} \frac{2s^2-s-2}{s-2}\, ds = 2\pi i\, f(2) = 2\pi i(8-4) = \underline{8\pi i}$ (with $f = 2s^2 - s - 2$)

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Panel 10

① If f is entire and $|f(z)| \leq M|z|$ then $f(z) = \underline{cz}$

HW

Hint: Pick z_0 and R and $C_R(z_0)$.

Note: $|f(z)| \leq M(R + |z_0|)$ (triangle inequality) if $z \in C_R(z_0)$

Then show that $f'' = 0$

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