Panel 1

Integration Theorems

Integral Estimation Theorem: f cont. on curve C and $|f(z)| \leq M$ on C. Then

How-To Theorem: f continuous in domain D. Then the following are equivalent:

a) f has antiderivative

b)

c)

1

Panel 2

Cauchy-Goursat Theorem: If f is analytic in a simply connected domain D and f is analytic in D then

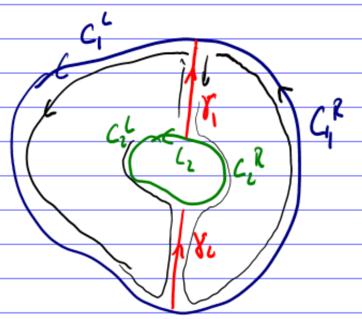
Deformation Theorem: If C_1 and C_2 are two simple closed curves, positively oriented, with C_1 inside C_2, and f analytic between them:

2

Panel 3

We proved Cauchy-Goursat using Green's Theorem if f' continuous.

Proof of Deformation Theorem: C_1 and C_2 closed simple curves with C_1 inside C_2, and f analytic between them



$$\int_{C_1^R - \gamma_1 - C_2^R - \gamma_2} f(z)\,dz = 0 \quad \text{by Cauchy Goursat}$$

$$\int_{C_1^L + \gamma_2 - C_2^L + \gamma_1} f(z)\,dz = 0$$

$$\int_{C_1^R - \cancel{\gamma_1} - C_2^R - \cancel{\gamma_2}} f\,dz + \int_{C_1^L + \cancel{\gamma_2} - C_2^L + \cancel{\gamma_1}} f\,dz = \int_{C_1 - C_2} f\,dz = 0 \iff \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

3

Panel 4

Complex Homework

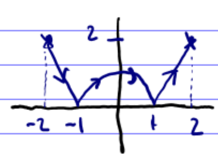
① Show that $\left| \int_C \frac{z}{z^2-1}\,dz \right| \le \frac{4\pi}{3}$ where C is the upper half of $|z|=2$

② Show that $\left| \int_C \frac{1}{z^4}\,dz \right| \le 4\sqrt{2}$ where C is the line from $z=i$ to $z=1$

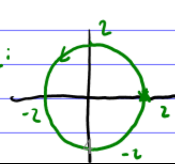
③ Evaluate (a) $\int_{C_1} z^2 + 3z\,dz$ (b) $\int_{C_2} \cos(z^2)\,e^z\,dz$

(c) $\int_{C_3} \frac{3}{z}\,dz$

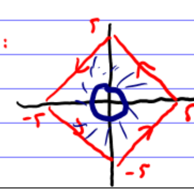
C_1:



C_2:



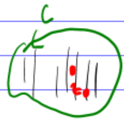
C_3:



Panel 5

Cauchy's Integration Formula: f analytic inside and on a simple, closed curve C (pos. oriented).

Then



$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \quad \forall z_0 \text{ inside } C$$

Ex:

$$\int_{|z|=2} \frac{\overbrace{z^2+3z+1}^{f(z)}}{z-1}\,dz = 2\pi i\, f(1) = 2\pi i\, 5 = 10\pi i$$

5

Panel 6

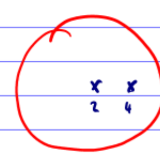
$$\int_{|z|=1} \frac{z}{(z-2)(z-4)}\,dz = 0$$

$$\int_{|z|=3} \frac{z}{(z-2)(z-4)}\,dz = 2\pi i\, f(2) = 2\pi i \frac{2}{-2} = -2\pi i$$

with $f(z) = \frac{z}{z-4}$

$$\int_{|z|=5} \frac{z}{(z-2)(z-4)}\,dz$$

PF Decomp.



6

Panel 7

$$\frac{z}{(z-2)(z-4)} = \frac{A}{z-2} + \frac{B}{z-4} = \frac{A(z-4)+B(z-2)}{(z-2)(z-4)}$$

$$z = A(z-4) + B(z-2) \qquad \forall z$$

pick $z=2$: $2 = A(-2) \Rightarrow A = -1$

pick $z=4$: $4 = B\,2 \Rightarrow B = 2$

$$\int_C \frac{z}{(z-2)(z-4)}\,dz = \int_C \frac{\overset{f(z)}{-1}}{z-2}\,dz + \int \frac{\overset{f(z)}{2}}{z-4}\,dz = -2\pi i + 2\cdot 2\pi i = \underline{\underline{2\pi i}}$$

$$2\pi i\, f(2) \quad + \quad 2\pi i\, f(4)$$

7

Panel 8

Proof of: $\displaystyle \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz = f(z_0)$

Def. theorem

① Look at $\displaystyle \int_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = \int_{C_R} \frac{f(z)-f(z_0)}{z-z_0}\,dz$ for any small R

② $\displaystyle \int_{C_R} \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{\cancel{z_0}+\cancel{R}\cancel{e^{it}}-\cancel{z_0}}\, i\cancel{R}\cancel{e^{it}}\,dt = i\int_0^{2\pi} dt = \underline{\underline{2\pi i}}$

circle, center z_0, radius R:

$z(t) = z_0 + Re^{it}$

$dz = iRe^{it}\,dt$

8

Panel 9

So $$\int_{C_R} \frac{f(z)-f(z_0)}{z-z_0}\,dz = \int_{C_R} \frac{f(z)}{z-z_0}\,dz - f(z_0)\int_{C_R} \frac{1}{z-z_0}\,dz =$$

$$= \int_{C_R} \frac{f(z)}{z-z_0}\,dz - f(z_0)\,2\pi i$$

I know f is continuous so: given any $\varepsilon > 0$ there is $\delta > 0$ st.

$|f(z)-f(z_0)| < \varepsilon$ if $\underbrace{|z-z_0| < \delta}$, circle radius δ

$$\Rightarrow \left| \int_{C_\delta} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| \le \int_{C_\delta} \frac{|f(z)-f(z_0)|}{|z-z_0|}\,dz \le \frac{\varepsilon}{\delta}\int_{C_\delta} dz = \frac{\varepsilon}{\cancel{\delta}}\cdot 2\pi \cancel{\delta} = 2\pi\varepsilon$$

$$\Rightarrow \left| \int_{C} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| = \left| \int_{C_\delta} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| < 2\pi\varepsilon \quad \forall \varepsilon$$

9

Panel 10

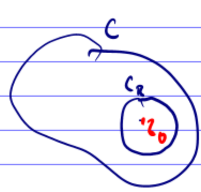
Thus: $$\int_{C} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$$

$$= \int_{C_R} \frac{f(z)}{z-z_0}\,dz - 2\pi i\, f(z_0)$$

$$\Rightarrow f(z_0) = \frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{C} \frac{f(z)}{z-z_0}\,dz$$

def. thm

q.e.d.



10

Panel 11

Cauchy's Int. Formula rephrased:

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz \quad \forall z_0 \in D$$

Replace z_0 by variable s:

$$f(s) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - s} dz \quad \text{is a function of } s$$

$$f'(s) = \frac{\partial}{\partial s} f(s) = \frac{1}{2\pi i} \int \frac{\partial}{\partial s} \frac{f(z)}{z-s} dz = \frac{1}{2\pi i} \int \frac{f(z)}{\underbrace{(z-s)^2}} dz$$

$$f''(s) = \frac{1}{2\pi i} \int \frac{f(z) \cdot 2}{(z-s)^3} dz, \quad f'''(s) = \frac{1}{2\pi i} \int \frac{1 \cdot 2 \cdot 3 \cdot f(z)}{(z-s)^4} dz$$

11

Panel 12

General Cauchy Integral Formula: f analytic inside and on a simple, closed curve C (pos. oriented). Then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w - z)^{\underline{n+1}}} dw$$

Ex: $$\int_{|z|=1} \frac{e^{2z}}{z^4} dz = \int_{|z|=1} \frac{e^{2z}}{(z-0)^4} dz = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{2\pi i}{6} \cdot 8 = \frac{8}{3} \pi i$$

$n = 3$

12

Panel 13

Consequences

(1) f analytic $\Rightarrow$ f' is analytic $\Rightarrow$ f'' analytic $\Rightarrow$ ....

Once analytic $\Rightarrow$ always analytic, all derivatives exist

in $\mathbb{R}$? f cont., not diff'ble $\Rightarrow$ $|x| \sim x^{1/3}$

f diff'ble once, but not twice: $x^{4/3}$

f diff'ble $2x$, but not $3x$: $x^{7/3}$

(2) Corollary: $f = u + iv$ analytic, then $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$.

In fact, u and v are inf. often diff'ble!

$f' = u_x + iv_x$, $f'' = u_{xx} + iv_{xx}$

13

Panel 14

Morera's Thm: If D is a domain and

$$\int_C f(z)\,dz = 0 \quad \forall \text{ closed curves in } D$$

then f is analytic!

$\Rightarrow$ f has anti derivative F and F is analytic

$\Rightarrow$ $F' = f$ is analytic

14

Panel 15

(4) Cauchy's Inequality: If f is analytic inside a circle C_R centered at z_0, and $|f(z)| \leq M$ on C_R, then

$$|f^{(n)}(z_0)| \leq \bigcirc$$

Easter Present: figure this out

Hint: Cauchy general. int. formula

Next: Fund. Thm. of Algebra: $p_n(z) = 0$ has n zeros

polyn. of degree n

15