Panel 1

Complex Analysis: Integration

$\int_\gamma z^2 dz$ where γ is curve from i to 1

$z(t) = i + t(1-i)$, $t \in [0,1]$, $dz = (1-i)dt$

$\int_\gamma z^2 dz = \int_0^1 (i+t(1-i))^2 (1-i)dt = \int_i^1 u^2 du = \frac{1}{3}u^3\Big|_i^1 = \frac{1}{3}(1^3 - i^3) = \frac{1}{3}(1+i)$

let $u = i + t(1-i)$

$du = (1-i)dt$

if $t=0 \Rightarrow u = i$

$t = 1 \Rightarrow u = 1$

1

Panel 2

Integral Estimation: If f is continuous along a curve γ then and $|f(z)| \le M$ on curve γ then:

$$\left| \int_\gamma f(z)\,dz \right| \le M \cdot \text{Length}(\gamma)$$

$z(t)$ is γ

Proof: $\left| \int_\gamma f(z) dz \right| = \left| \int_a^b f(z(t))\, z'(t)\, dt \right|$

$\le \int_a^b |f(z(t))|\,|z'(t)|\,dt \le$

details from book

$\le M \int_a^b |z'(t)|dt = M \int_a^b \sqrt{(x')^2 + (y')^2}\, dt =$

$= M \text{ length}(\gamma)$

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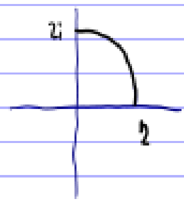
Panel 3

Estimate $\int_\gamma \frac{z+4}{z^3-1}\,dz$, γ is first $\frac{1}{4}$ of $|z|=2$



Need: $|f(z)| \le M$, length$(\gamma) = \pi$

z is on the curve, so that $|z|=2$

$\Rightarrow |z+4| \le |z| + 4 = 6$

$|z^3-1| \ge |z|^3 - 1 = 7$ $\Rightarrow \left|\frac{z+4}{z^3-1}\right| \le \frac{6}{7}$

$\Rightarrow \left| \int_\gamma \frac{z+4}{z^3-1}\,dz \right| \le \frac{6\pi}{7}$

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Panel 4

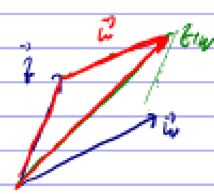
Basic Inequalities:

$|z+w| \le |z| + |w|$ triangle inequality

Handy for later

$|z-w| \ge |w| - |z|$

$|z| - |w|$



$|w| = |z + (w-z)| \le |z| + |w-z|$

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Panel 5

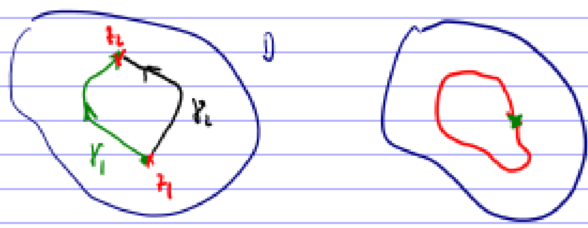
Thm: Suppose f is continuous in a domain D. Then the following are equivalent:

a) f has an anti derivative throughout D, called F

b) If γ_1, γ_2 are two curves in D from z_1 to z_2 then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$$

c) If γ is any closed path in D then $\int_{\gamma} f(z)\,dz = 0$



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Panel 6

Ex: $\int_{\gamma} z^2\,dz$, γ line from i to 1.

z^2 has anti derivative $F(z) = \frac{1}{3}z^3$ everywhere

$$\int_{\gamma} z^2\,dz = \frac{1}{3}z^3\Big|_i^1 = \frac{1}{3}(1+i)$$

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Panel 7

Ex: Evaluate $\int_\gamma \frac{1}{z^2}\,dz$ for $|z|=R$ ← circle radius R or $z(t)=Re^{it}, t\in[0,2\pi]$

hard: $\int_\gamma \frac{1}{z^2}dz = \int_0^{2\pi} \frac{1}{[Re^{it}]^2} Rie^{it}\,dt = \frac{1}{R}\int_0^{2\pi} i\,e^{-2it}e^{it}\,dt =$

$= \frac{1}{R}\, i \int_0^{2\pi} e^{-it}\,dt = \frac{1}{R}\, i \left(-\frac{1}{i}\right) e^{it}\Big|_0^{2\pi} = -\frac{1}{R}\left[e^{2\pi i}-1\right]$
$= 0$

easy: $1/z^2$ has antiderivative $-1/z$ on the circle

$\Rightarrow \int_\gamma 1/z^2\,dz = 0$ for any closed curve γ

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Panel 8

Ex: Evaluate $\int_\gamma \frac{1}{z}\,dz$ for $|z|=R$

~~Easy~~: $1/z$ has anti deriv. NOT on full circle $\text{Log}(z)$ $\Rightarrow \int_{|z|=R} 1/z\,dz = 0$

Hard: $z=Re^{it}$, $\int_0^{2\pi} \frac{1}{\cancel{Re^{it}}} \cdot \cancel{R}i\cancel{e^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i$

WHAT GIVES?

Problem: $\text{Log}(z)$ can't be analytic on $|z|=R$

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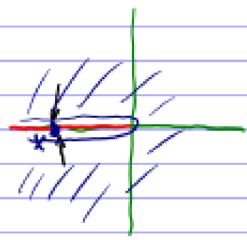
Panel 9

Thm: Log(z) is not analytic for $\{Re(z) \le 0\}$



$Log(z) = \ln(|z|) + i\,Arg(z)$

take x on neg. real axis:

approach x from above: $Log(z) = \ln(|x|) + i\pi$

approach x from bottom: $Log(z) = \ln(|x|) + i(-\pi)$

$\Rightarrow$ Log(z) ain't continuous on $\{x < 0\}$

Bummer: Redefine $Log(z) = \ln(|z|) + i\,Arg(z)$, $|z| \neq 0$, $-\pi < Arg(z) \le \pi$

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Panel 10

Theorem: C is a closed, simple (non-intersecting) curve and f is analytic inside and on C. Also, f' is continuous there. Then:

$$\int_C f(z)\,dz = 0$$

Recall Green's Theorem: If P(x,y) and Q(x,y) are continuous in D, and all partials exist and are continuous then:

$$\int_C P\,dx + Q\,dy = \iint_R Q_x - P_y\,dA$$

$$\int_C P(x,y)\,dx = \int_C P(x(t), y(t))\,x'(t)\,dt$$

$$\int_C Q(x,y)\,dy = \int_C Q(x(t), y(t))\,y'(t)\,dt$$

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Panel 11

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt =$$

$$\int_a^b \big(u(x(t),y(t)) + i\,v(x(t),y(t))\big)\big(x'(t) + i\,y'(t)\big)\,dt$$

$$\int_a^b u\cdot x' - v y'\,dt + i\int u\cdot y' + v x'\,dt$$

$$\int_a^b u\,dx - v\,dy + i\int u\,dy + v\,dx$$

Green: $$\iint_D -v_x - u_y\,dA + i\iint_D u_x - v_y\,dA = 0$$

by CR equations! #

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Panel 12

Cauchy: f analytic + f' continuous inside and on C

Then: $\int_C f(z)\,dz = 0$ $\forall$ simple, closed curves C

Goursat realized that f' is not necessary.

Cauchy-Goursat Thm: If f is analytic inside and on C

then $\int_C f(z)\,dz = 0$ $\forall$ simple, closed curves C

(Proof later)

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Panel 13

Augustin-Louis Cauchy

Augustin-Louis Cauchy around 1840./ Lithography of Zéphirin Belliard after a painting by Jean Roller.

Born	21 August 1789 Paris, France
Died	23 May 1857 (aged 67) Sceaux, France
Residence	France
Nationality	French
Fields	Infinitesimal calculus Complex analysis

Edouard Goursat

Edouard Goursat

Born	21 May 1858 Lanzac, Lot
Died	25 November 1936
Nationality	France
Fields	mathematics
Alma mater	École Normale Supérieure
Doctoral advisor	Gaston Darboux

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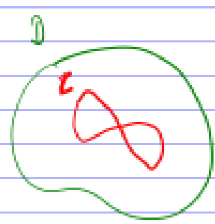
Panel 14

Cauchy - Goursat Theorem (Improved) no holes

If f is analytic in a simply connected domain D. Then

$$\oint_C f(z)\,dz = 0 \quad \forall\ \cancel{\text{simple}} \text{ closed curves } C \text{ in } D$$

Corollary: If f is analytic in a simply connected domain D then f has an antiderivative!



If f analytic in D

$$\Rightarrow \oint_C f(z)\,dz = 0$$

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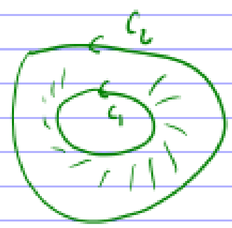
Panel 15

Cauchy - Goursat Theorem

If f is analytic in a simply connected domain D then

$$\int_C f(z)\,dz = 0 \quad \text{for every closed curve } C \text{ in } D$$

Corollary: If C_1 and C_2 are two simple closed curves, positively oriented, with C_1 inside C_2. Then if f is analytic between C_1 and C_2 then



$$\int_{C_1} f\,dz = \int_{C_2} f\,dz$$

Deformation theorem!

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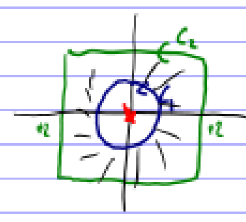
Panel 16

Ex: Find $\int_C e^{z^3}\,dz = 0$ where C is square of length 4.

square closed and e^{z^2} is analytic inside!

Ex: Find $\int_C \frac{1}{3z}\,dz$ where C square of side 4.

old-fashioned: 4 integrals over straight lines;



$$\int_{\text{square}} \frac{1}{3z}\,dz = \int_{\text{circle}} \frac{1}{3z}\,dz = \frac{1}{3}\int \frac{1}{z}\,dz = \frac{1}{3}\,2\pi i = \underline{\underline{\frac{2\pi}{3}i}}$$

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