Panel 1

Last Time:

Various computations: $2^i, i^2, i^i, i^{i^i}$

Complex parametric functions:

$$z(t) = x(t) + i y(t) \quad \text{e.g. } z(t) = 1 + (3-i)t^2$$

Derivatives and Integrals of

$$\frac{d}{dt} z(t) = x'(t) + i y'(t)$$

$$\int z(t)\,dt = \int x(t)\,dt + i \int y(t)\,dt$$

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Panel 2

So we worked out two possibilities (principle part only)

$$i^{i^i} = \begin{cases} (i^i)^i = -i = 1/i & \text{A} \\ \underset{i^{(i^i)}}{} e^{i\pi/2 e^{-\pi/2}} = \cos(\frac{\pi}{2}e^{-\pi/2}) + i \sin(\pi/2\, e^{-\pi/2}) & \text{B} \end{cases}$$

Which one is right? Well, think about 3^{3^3}...

A $(3^3)^3 = 27^3$

B $3^{(3^3)} = 3^{27}$ ✓

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Panel 3

Complex Homework:

① Find the real and imaginary (principle) parts of:

a) π^i b) i^π

② Find the magnitude (abs. value) as a decimal # for:

a) $\cos(3i)$ b) $\sin(3i)$

What is interesting about your answer?

③ Suppose $z(t) = (5+i) + 7t$ and $w(t) = 3ie^{2it}$. Find

a) $z'(t)$ b) $w'(t)$ c) $\int_0^1 z(t)\,dt$ d) $\int_0^\pi w(t)\,dt$

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Panel 4

Ex: $\int_0^\pi e^{it}\,dt \quad \cdot \int_0^\pi x(t) + iy(t)\,dt$

① $\int_0^\pi \cos(t) + i\sin(t)\,dt = \sin(t) - i\cos(t)\Big|_0^\pi$

② $\int_0^\pi e^{it}\,dt = \frac{1}{i}e^{it}\Big|_0^\pi = \frac{1}{i}(e^{\pi i} - e^0) = -i((-1)-1) = 2i$

$= -i(\cos(t) + i\sin(t))\Big|_0^\pi = \sin(t) - i\cos(t)\Big|_0^\pi$

$= \sin(\pi) - \sin(0) - i(\cos(\pi) - \cos(0))$

$-i(-1-1) = 2i$

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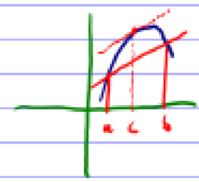
Panel 5

Most rules from diff. and int. apply but not all.

Neither the Mean Value Thm. for Differentiation

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

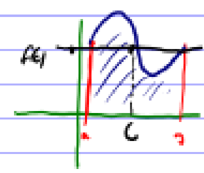


Find counter example

$z(t)$

nor the Mean Value Thm for Integration

$$\int_a^b f(t)\,dt = f(c)(b-a)$$



Find counter example

$z(t)$

apply for $z(t) = x(t) + iy(t)$

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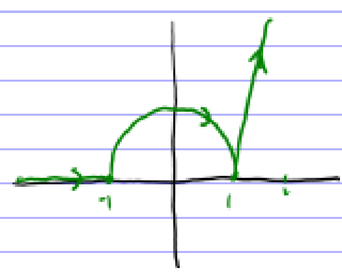
Panel 6

Move about Paths:

① Find a line segment from A to B $\quad z(t) =$

② Find upper half of circle centered at i with radius 1 $\quad z(t)$

③ Parametrize the following curve



$z(t) =$

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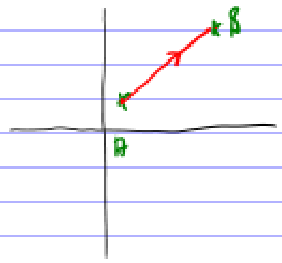
Panel 7

More about Paths:

① Find a line segment from A to B



$\cancel{y = mx + b}$

$$\boxed{z(t) = A + t(B - A) \quad , t \in [0,1]}$$

$z(0) = A, \; z(1) = A + B - A = B$

Ex: line from 1 to $3-i$: $z(t) = 1 + t(3 - i - 1) = 1 + t(2 - i)$

$t \in [0,1]$

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Panel 8

More about Paths:

① Find a line segment from A to B

$z(t) = A + t(B - A) \;, t \in [0,1]$

② Find upper half of circle centered at i with radius 2

$z(t) = 2e^{it} + i \qquad , t \in [0, \pi]$ ☺

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Panel 9

More about Paths:

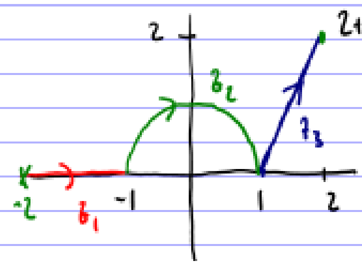
① Find a line segment from A to B

$z(t) = A + t(B - A)$

② Find upper half of circle centered at i with radius 1

$z(t) = e^{it} + i \quad , t \in [0, \pi]$

③ Parametrize the following curve



$z_1(t) = -2 + t(-1+2) \quad , t \in [0,1]$

$= -2 + t$

$z_2(t) = e^{it} \quad , t \in [\pi, 0]$

$z_3(t) = 1 + t(2+2i-1) \quad , t \in [0,1]$

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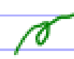
Panel 10

Definitions:

A curve (arc) is the graph of $z(t) = x(t) + iy(t)$, x, y are continuous

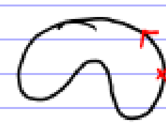
A simple curve (arc) is a non-intersecting curve

 not simple

A simple, closed curve: simple with start = end

A Jordan curve: a simple, closed curve

A Jordan curve with positive orientation: as you walk along curve, inside is to the left.



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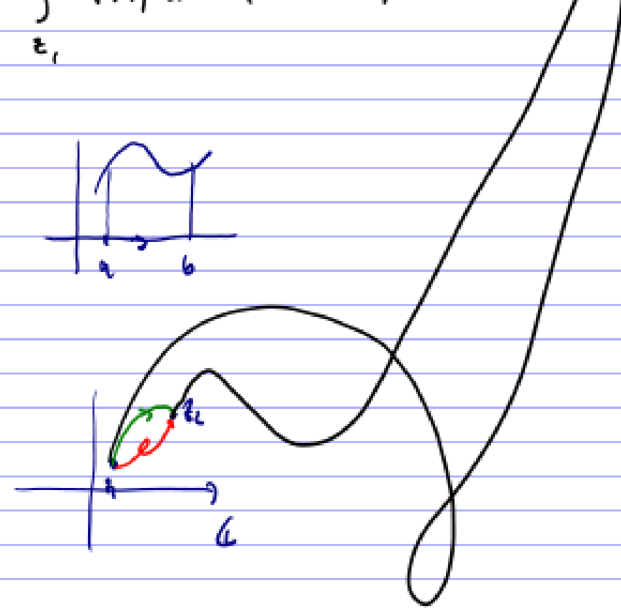
Panel 11

How to define $\int_{z_1}^{z_2} f(z)\,dz$, $f(z)$ complex function

$$\int_a^b f(x)\,dx$$



$$\int_{z_1}^{z_2} f(z)\,dz$$

Panel 12

Contour Integral:

gamma

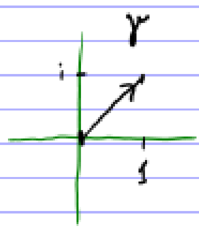
Suppose $z_1, z_2 \in \mathbb{C}$ and $\gamma(C)$ is a curve from z_1 to z_2 with some parametrization $z(t)$, $t \in [a,b]$, and $z_1 = z(a)$ and $z_2 = z(b)$. Then

$$\int_{\gamma(C)} f(z)\,dz = \int_a^b f(z(t)) \frac{dz}{dt}\,dt = \int_a^b f(z(t))\, z'(t)\,dt$$

Panel 13

<u>Ex:</u> Let γ be the straight line from $z_0 = 0$ to $z_1 = 1+i$.

Compute $\displaystyle\int_\gamma z^2 \, \underline{dz} = \int_0^1 \left(t(1+i)\right)^2 z'(t)\,dt = \int_0^1 t^2(1+i)^2(1+i)\,dt$



$\gamma: z(t) = 0 + t(1+i-0) = \underline{t(1+i)},\ t \in [0,1]$

$$= \int_0^1 (1+i)^3 t^2 \, dt = (1+i)^3 \frac{1}{3} t \Big|_0^1 =$$

$$= \underline{\underline{\frac{1}{3}(1+i)^3}}$$

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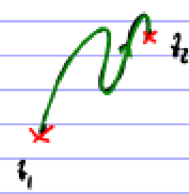
Panel 14

<u>Natural questions:</u> $\displaystyle\int_\gamma f(z)\,dz$



<u>Q: 2</u> If γ_1 and γ_2 are two paths from z_1 to z_2, do you get the same answer?

<u>Q: 1</u> If you have 2 parametrizations of the same curve, do you get the same answer?

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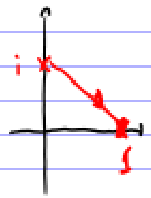
Panel 15

Ex: Graph $z_1(t) = i + t(1-i)$, $t \in [0,1]$ and

$z_2(t) = i + 2t(1-i)$, $t \in [0, \frac{1}{2}]$

Find $$\int_{z_1(t)} z\,dz = \int_a^b z(t)\, z'(t)\,dt = \int_0^1 [i + t(1-i)](1-i)\,dt$$



$$\int_{z_2(t)} z\,dz = \int_0^{1/2} [i + 2t(1-i)]\, 2(1-i)\,dt =$$

$2t = u$, $dt = \frac{1}{2}du$

$$\int_0^1 [i + u(1-i)]\, \cancel{2}(1-i)\cancel{\tfrac{1}{2}}\,du$$

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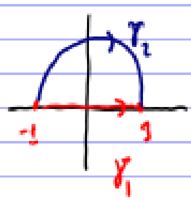
Panel 16

Ex: Evaluate $\int_{\gamma_1} z^2\,dz$ and $\int_{\gamma_2} z^2\,dz$



γ_1: $z(t) = -1 + t(1-(-1)) = -1 + 2t$, $t \in [0,1]$

$$\int_{\gamma_1} z^2\,dz = \int_0^1 (-1+2t)^2\, 2\,dt = 2\int_0^1 1 - 4t + 4t^2\,dt =$$
$$= 2\left(t - 2t^2 + \tfrac{4}{3}t^3\right)\Big|_0^1 = 2\left(1 - 2 + \tfrac{4}{3}\right) = \tfrac{2}{3}$$

γ_2: $z(t) = e^{it}$, $t \in [\pi, 0]$: $$\int_{\gamma_2} z^2\,dz = \int_\pi^0 (e^{it})^2\, i e^{it}\,dt = i\int_\pi^0 e^{3it}\,dt =$$
$$= \tfrac{1}{3} e^{3it}\Big|_\pi^0 = \tfrac{1}{3}\left(1 - e^{3\pi i}\right) = \tfrac{2}{3}$$

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Panel 17

Ex: Evaluate $\int_{\gamma_1} \bar{z}\, dz$ and $\int_{\gamma_2} \bar{z}\, dz$

same curves γ_1, γ_2 !

HW

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Panel 18

What is the definition of:

- A complex number
- Adding and Multiplying, Sub and Div, graphically
- Complex roots
- Mapping properties of complex functions
- Arg(z) and arg(z)
- The limit of a complex function f(z) as z approaches c is L
- Continuity of a complex function f(z) at a point z = c
- The complex derivative of a function f(z)
- Analytic function
- CR equations
- Entire function
- Harmonic conjugate of a function u
- Harmonic function
- e^z, sin(z), cos(z), log(z), and Log(z)
- Euler's Formula
- Functions z(t), its integral and derivative
- Different paths (line segments and circles)
- Contour Integrals

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