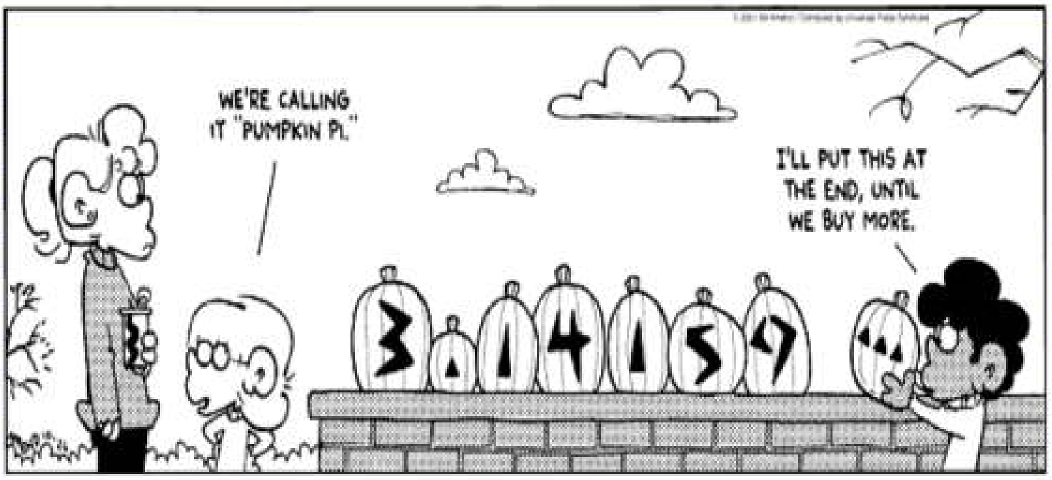
Panel 1

Pi Day was Yesterday

3.14 22/7

14/3



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Panel 2

http://www.realpiday.org/

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Panel 3

Complex Concepts

Limits: $\lim_{z \to z_0} f(z) = L$ ← tricky but like in $\mathbb{R}^2$

Continuity: $\lim_{z \to z_0} f(z) = f(z_0)$

$\mathbb{C}$-Diff'ble: $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ z's - diff'ble; $\bar{z}$'s - not diff'ble

Analytic $\mathbb{C}$-diff'ble in $D_\varepsilon(z_0)$ (i.e. nbhd of z_0)

Harmonic $u_{xx} + u_{yy} = 0$

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Panel 4

CR: f is diff'ble ⇔ $u_x = v_y$

$u_y = -v_x$

f analytic ⇒ u, v are harmonic ($\mathrm{Re}(f) = u$)

If u is harmonic then there is a harmonic conjugate v s.t. $f = u + iv$ is analytic

Theorems:

f analytic & $f'(z) \equiv 0$

f analytic, $|f(z)| = c$

f & $\bar{f}$ both analytic

} f must be constant.

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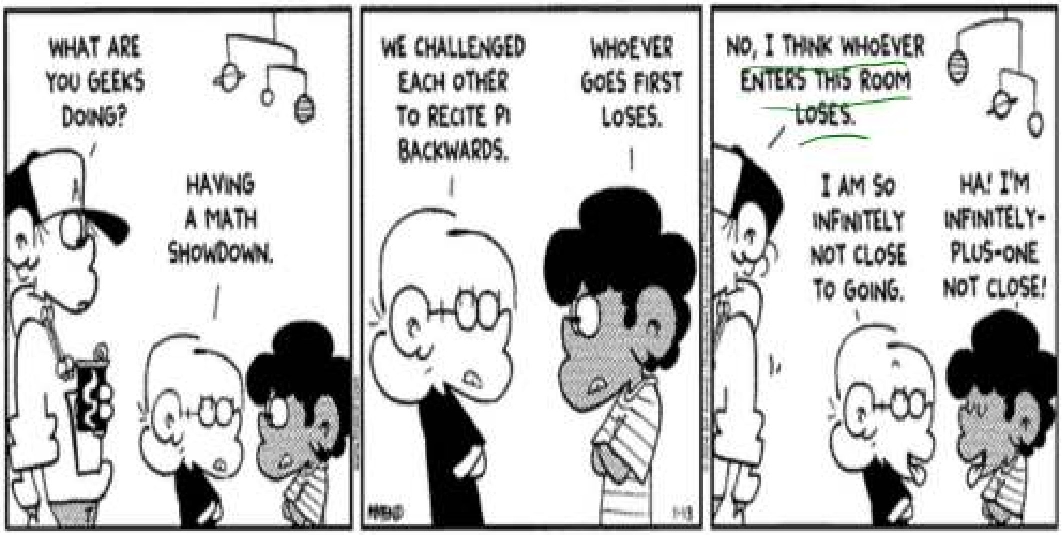
Panel 5



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Panel 6

Complex HW

① Show that the following functions are entire:

a) $f(z) = 3x + y + i(3y - x)$ ✓

b) $f(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$ ✓

② Show that the following functions are nowhere analytic

a) $f(z) = xy + iy$ ✓

b) $f(z) = e^y e^{ix}$ ✓

③ Suppose $f(z)$ is analytic in a domain D and $f(z)$
Ⓔ is real-valued. Prove that $f(z)$ must be constant.

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Panel 7

Suppose $f(z)$ is analytic in a domain D and $f(z)$ is real-valued. Prove that $f(z)$ must be constant.

$f = u + iv$, $v = 0$

$\Rightarrow v_x = v_y = 0$

$u_x = v_y = 0$

$f'(z) = u_x + i v_x = 0 \Rightarrow f$ is const

Same: f is analytic + purely imaginary $\Rightarrow f$ is const.

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Panel 8

Complex HW

① Show that $u(x,y)$ is harmonic and find harm. conjugate:

a) $u(x,y) = 2x - x^3 + 3xy^2$

b) $u(x,y) = \sinh(x)\sin(y)$

c) $u(x,y) = \frac{y}{x^2+y^2}$

② Prove that if $v(x,y)$ and $V(x,y)$ are both harmonic conjugates of $u(x,y)$ then v and V can differ by at most a constant.

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more ⇒

Panel 9

Find harmonic conjugate for $u(x,y) = \frac{y}{x^2+y^2}$

① $u(x,y) = y \cdot (x^2+y^2)^{-1}$ is harmonic.

$u_x = -2yx(x^2+y^2)^{-2}$

$u_{xx} = -2y(x^2+y^2)^{-2} + 4xy \cdot 2x(x^2+y^2)^{-3}$

$u_y = (x^2+y^2)^{-1} - 2y^2(x^2+y^2)^{-2}$

u_{yy} give up!

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Panel 10

Find harmonic conjugate for $u(x,y) = \frac{y}{x^2+y^2}$

Know: $u_x(x,y) = \frac{-2xy}{(x^2+y^2)^2}$ and $u_y(x,y) = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}$

$u_x = v_y = -\frac{2xy}{(x^2+y^2)^2} \Rightarrow v = -2x\int \frac{y}{(x^2+y^2)^2}\,dy =$

$v = -2x\left(-\frac{1}{2}\right)(x^2+y^2)^{-1} = \frac{x}{x^2+y^2} + C(x)$

$x(x^2+y^2)^{-1}$

$-u_y = v_x = \frac{1}{x^2+y^2} - 2x^2(x^2+y^2)^{-2} + C'(x)$

$\Rightarrow C'(x) = 0$

$= \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + C'(x) =$

$= \frac{x^2+y^2}{(x^2+y^2)^2} - \frac{2x^2}{(x^2+y^2)^2} + C'(x) = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} + C'(x) = -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} + C'(x)$

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Panel 11

Find harmonic conjugate for $u(x,y) = \frac{y}{x^2+y^2}$

$v(x,y) = \frac{x}{x^2+y^2}$ is harm. conj. to $u(x,y)$

$\Rightarrow f(z) = \frac{y}{x^2+y^2} + i\frac{x}{x^2+y^2}$ is analytic, also u, v are harmonic!

$$= \frac{y+ix}{x^2+y^2} = \frac{i(-iy+x)}{z\bar{z}} = \frac{i\bar{z}}{z\bar{z}} = \underline{\underline{i/z}}$$

$f'(z) = -i/z^2 = u_x + iv_x$

check (or not)

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Panel 12

Prove that if $v(x,y)$ and $V(x,y)$ are both harmonic conjugates of $u(x,y)$ then v and V can differ by at most a constant.

Since $v(x,y)$ is harmonic conj of $u(x,y)$ then

$u_x = v_y \quad u_y = -v_x$ $\Rightarrow f = u + iv$ is analytic

& $V(x,y)$ is h. conj of $u(x,y)$

then $u_x = V_y$; $u_y = -V_x$ $\Rightarrow F = u + iV$ is analytic

$\Rightarrow v_x = v_y = V_y$ $f - F$ is analytic and purely imag.

If only know u_x $\Rightarrow u_x = v_y$ $u_x = V_y$

Then $v = \int v_y \, dy$ $V = \int V_y \, dy$

$\Rightarrow f - F$ are const $\Rightarrow v - V = \text{const}$

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Panel 13

$V = \int V_y \, dy$ and $V = \int V_y \, dy$

$= \_\_ + c_1(x)$ $= \_\_ + c_2(x)$

$U_y = -V_x + c_1'(x)$ $U_y = -V_x + c_2'(x)$

$\Rightarrow c_1'(x) = 0$ $\Rightarrow c_2'(x) = 0$

Therefore V and v differ by a constant

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Panel 14

We defined special functions:

$e^z = e^x e^{iy} = e^x (\cos(y) + i \sin(y))$

$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$

$\sin(z) = \frac{1}{2i}(e^{iz} - e^{iz})$

analytic, periodic, unbounded

$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$

$\log(z) = \ln(|z|) + i(\theta + 2\pi k)$, θ principle angle ← multi-valued

$\mathrm{Log}(z) = \ln(|z|) + i\theta$

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Panel 15

③ We defined $e^z = e^x e^{iy} = e^x(\cos(y) + i\sin(x))$

a) solve $e^z = 1 - i$ and $e^z = -2$ (all solutions)

b) show that e^z is entire but $e^{\bar{z}}$ is nowhere diff'ble

c) show that $|e^{-2z}| < 1$ iff $Re(z) > 0$

④ We defined $\log(z) = \ln(r) + i(\theta + 2k\pi)$ and

$Log(z) = \ln(r) + i\theta$, where $z = re^{i\theta}$

a) find $\log(i)$ and $\log(-3)$

b) show that $Log(i^3) \neq 3\,Log(i)$ but

$Log(1+i)^2 = 2\,Log(1+i)$

More $\Rightarrow$

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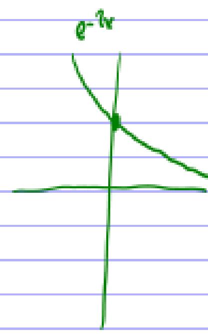
Panel 16

$$\underline{|e^{-2z}|} = |e^{-2x-2iy}| = |e^{-2x}e^{-2iy}| = e^{-2x}|e^{-2iy}| = e^{-2x}$$

$e^z = e^x e^{iy}$



$\underline{e^{-2x} < 1}$

iff $\underline{x > 0}$

$Re(z) > 0$

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Panel 17

(5) For extra credit, try to determine the values of

a) 2^i b) i^i c) i^{i^i}

$i^2 = -1$

Calc: $\frac{d}{dx}(x^x)$

$2^i = e^{\log(2^i)}$

$= e^{i\ln(2)} = \underline{\cos(\ln(2)) + i\sin(\ln(2))}$

$x^x = e^{\ln(x^x)} = e^{x\ln(x)}$

Note: $|5^i| = 1$

$\frac{d}{dx} e^{x\ln(x)} = \underbrace{e^{x\ln(x)}} \cdot [\ln(x) + x \cdot \frac{1}{x}]$

$|(11)^i| = 1$

$= \underline{x^x(\ln(x)+1)}$

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Panel 18

$i^i = e^{\log(i^i)} = e^{i\log(i)} = e^{i(i(\pi/2+2\pi k))} = e^{-\pi/2 - i 2\pi k} = e^{-\pi/2} \cdot e^{-2\pi k}$

principle, multi values

$\log(z) = \ln(|z|) + i(\theta + 2\pi k)$

$\log(i) = \ln(|i|) + i(\pi/2 + 2\pi k)$

$= i(\pi/2 + 2\pi k)$

$i^i \in \mathbb{R}$

$i^{(i)} = i^{i \cdot i} = i^{-1} = 1/i \leftarrow$ compute other way $i^{e^{-\pi/2}} = ?$

$= \frac{1}{i}$

$2^{2^2} = 2^4$

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Panel 19

Chapter 4: Integrals

Integration in Complex Analysis is very important and leads to beautiful, profound theorems with elegant proofs.

Pure mathematics is, in its way, the poetry of logical ideas. ~Albert Einstein

Before we integrate....

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Panel 20

Complex Parametric Functions

Def: A parametric function $r(t) = (x(t), y(t))$ is a function from $\mathbb{R}$ to $\mathbb{R}^2$. A complex parametric function $w(t) = x(t) + i\,y(t)$ is a function from $\mathbb{R} \to \mathbb{C}$ $(\sim \mathbb{R}^2)$

Ex: $z(t) = (3+i) + (1-i)t = x(t) + i\,y(t)$, $x(t) = 3+t$ $\quad t = x-3$

$y(t) = 1 - t = 1 - x + 3 = 4 - x$

$z(t) = \underbrace{\cos(t)}_{x} + i\underbrace{\sin(t)}_{y} = e^{it}$

Their graphs are curves in $\mathbb{R}^2$ or $\mathbb{C}$

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Panel 21

Derivatives of Complex parametric functions:

Def: If $z(t) = x(t) + i y(t)$ then $\frac{dz}{dt} = \frac{dx}{dt} + i \frac{dy}{dt} = x'(t) + i y'(t)$

Ex: If $z(t) = 5\cos(3t) + 5i\sin(3t)$, find $z'(t)$

$$z'(t) = -15\sin(3t) + 15i\cos(3t) = 15i(i\sin(3t) + \cos(3t)) =$$

Note: $z(t) = 5e^{i3t} \Rightarrow z'(t) = 15ie^{i3t} \Rightarrow 15ie^{i3t}$

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Panel 22

Integrals of Complex Parametric Functions

Def: If $z(t) = x(t) + i y(t)$ then

$$\int z(t)\,dt = \int x(t)\,dt + i\int y(t)\,dt$$

Ex: $\int_0^1 (1+it)^2\,dt = \int_0^1 1 - t^2\,dt + \int_0^1 2t\,dt$

$$= t - \frac{1}{3}t^3\Big|_0^1 + i\,t^2\Big|_0^1 = \# + i\#$$

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