Panel 1

Last Time

CR equations in rectangular and polar coordinates:

$$u_x = v_y \qquad r u_r = v_\theta$$
$$u_y = -v_x \qquad u_\theta = -r v_r$$

Analytic Functions: $\mathbb{C}$-diff'ble in a nbhd. of z_0

Entire functions: analytic everywhere in $\mathbb{C}$

Theorems: f analytic, $f' = 0 \Rightarrow f = \text{constant}$

$f, \bar{f}$ both analytic $\Rightarrow f$ const.

f analytic, $|f|$ const $\Rightarrow f$ const.

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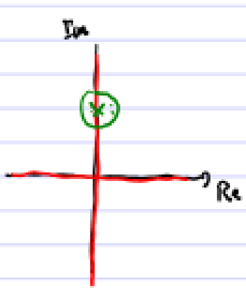
Panel 2

Ex: Is $f(z) = \underbrace{x^3 + 3xy^2}_{u} + i(\underbrace{y^3 + 3x^2y}_{v})$ diff'ble anywhere?
Is it analytic anywhere?

$$u_x = 3x^2 + 3y^2 = v_y = 3y^2 + 3x^2$$

$$u_y = 6xy = -v_x = 6xy$$

f is $\mathbb{C}$-diff'ble if $6xy = -6xy \iff$ if $x = 0$ or $y = 0$



but nowhere analytic!

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Panel 3

Complex HW

① Show that the following functions are entire:

a) $f(z) = 3x + y + i(3y - x)$

b) $f(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

② Show that the following functions are nowhere analytic

a) $f(z) = xy + iy$

b) $f(z) = e^y e^{ix}$

③ Suppose $f(z)$ is analytic in a domain D and $f(z)$ is real-valued. Prove that $f(z)$ must be constant.

CR $\quad f(z) = u(x,y) + i \cdot 0$

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Panel 4

Next we want to investigate the connection between analytic functions $f(z)$ and the component functions u, v

Def. A function $u(x,y)$ is called harmonic if all first + second derivatives are continuous and

$$u_{xx} + u_{yy} = 0$$

in a domain D

Note: $\Delta u = u_{xx} + u_{yy}$ is called Laplacian

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Panel 5

Ex: Which of these are harmonic functions:

$u(x,y) = x^3 - 3xy^2$

$u_x = 3x^2 - 3y^2 \qquad u_y = -6xy$ YES

$u_{xx} = 6x \qquad u_{yy} = -6x \quad \Rightarrow u_{xx} + u_{yy} = 0$ ✓

$v(x,y) = 3x^2y + y^3$

NO

$T(x,y) = e^{-y}\sin(x)$ –

$u_x = e^{-y}\cos(x) \qquad u_y = -e^{-y}\sin(x)$ YES

$u_{xx} = -e^{-y}\sin(x) \qquad u_{yy} = e^{-y}\sin(x) \quad \Rightarrow u_{xx} + u_{yy} = 0$

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Panel 6

Theorem: If f is analytic in a domain D and $f(z) = u(x,y) + iv(x,y)$, u, v have all 2^{nd} order derivatives continuous, then u and v are harmonic.

Proof: Need $u_{xx} + u_{yy} = 0$

$u_x = v_y \quad \Rightarrow u_{xx} = v_{yx}$ — order of diff. does not matter (Calc 3)

$u_y = -v_x \quad \Rightarrow u_{yy} = -v_{xy} = -v_{yx}$

$\Rightarrow u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0$ #

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Panel 7

Def: If $u(x,y)$ is harmonic and $v(x,y)$ a second harmonic function s.t. $f(z) = u(x,y) + i\,v(x,y)$ is analytic, then v is called harmonic conjugate of u.

Ex: $f(z) = z^2 = \underbrace{x^2 - y^2}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$

u and v are harmonic conjugates

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Panel 8

Thm: If $u(x,y)$ is harmonic in a 'special' domain D (simply connected), then it has a harmonic conjugate

Ex: $u(x,y) = \underline{x^3 - 3xy^2}$ is harmonic. Find harmonic conjugate!

$(= z^3)$ "cont'n"

Need v s.t. $u + iv$ is analytic: $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3) + C$

$u_x = 3x^2 - 3y^2 = v_y$

$\Rightarrow \underline{v} = \int v_y\, dy = \int 3x^2 - 3y^2\, dy =$

$= \underline{3x^2 y - y^3 + C(x)}$

$u_y = -6xy \qquad v_x = 6xy + C'(x)$

$\Rightarrow C'(x) = 0 \Rightarrow v(x,y) = 3x^2 y - y^3 + c$

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Panel 9

Ex: $u(x,y) = e^{-y}\sin(x)$. Find $v(x,y)$ such that $f(z) = u(x,y) + i\,v(x,y)$ is analytic. Find $f'(z)$

u is harmonic ✓

$u_x = e^{-y}\cos(x) = v_y \Rightarrow v(x,y) = -e^{-y}\cos(x) + C(x)$

$u_y = -e^{-y}\sin(x) \quad - \quad v_x = e^{-y}\sin(x) + C'(x)$

$\Rightarrow C'(x) = 0 \Rightarrow C(x) = \text{const.}$

$f(z) = e^{-y}\sin(x) + i\left(-e^{-y}\cos(x)\right) + C$

combine

$f'(z) = u_x + i v_x = e^{-y}\cos(x) + i\,e^{-y}\sin(x) = e^{-y}\left(\cos(x) + i\sin(x)\right) = e^{-y}e^{ix}$

$= e^{-y+ix} = e^{iz}$

$\Rightarrow f(z) = \frac{1}{i}e^{iz} = \underline{-i\,e^{iz}}$

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Panel 10

Chapter 3: Exp. and Friends

Def: $e^z = e^{x+iy} = e^x e^{iy} = \underbrace{e^x\left(\cos(y)\right.}_{u} + \overbrace{\left. i\sin(y)\right)}^{v}$

Properties

① $|e^z| = |e^x e^{iy}| = e^x |e^{iy}| = e^x$

② $e^z \neq 0$

③ e^z is periodic with period $2\pi i$, because
$e^{z+2\pi i} = e^{x+iy+2\pi i} = e^x e^{i(y+2\pi)} = e^x e^{iy} e^{i2\pi} = e^x e^{iy} = e^z$

④ $\frac{d}{dz}e^z = e^z$ (use CR and $f' = u_x + i v_x$) ← HW

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Panel 11

Ex: Solve $e^z = 1+i$

$$e^z = \sqrt{2}e^{i\pi/4} \Leftrightarrow e^x e^{iy} = \sqrt{2}e^{i\pi/4} \quad |\;|^{\text{abs}}$$

$$|e^x e^{iy}| = |\sqrt{2}e^{i\pi/4}|$$

$$e^x = \sqrt{2} \Rightarrow x = \ln|\sqrt{2}| = \frac{1}{2}\ln|2|$$

$e^z = 1+i \Rightarrow$

$$e^{\frac{1}{2}\ln|2|}e^{iy} = \sqrt{2}e^{i\pi/4}$$

$z = \frac{1}{2}\ln|2| + i\left(\frac{\pi}{4} + 2k\pi\right),\ k \in \mathbb{Z}$

$$\cancel{\sqrt{2}}e^{iy} = \cancel{\sqrt{2}}e^{i\pi/4}$$

$$y = \pi/4 + 2\pi k$$

$e^z = 2$

$z = \ln|2| + i\,2k\pi,\ k = 0, \pm 1, \pm 2, \ldots$

Panel 12

Def: If $z = re^{i\theta}$, define $\log(z) = \ln|r| + i(\theta + 2k\pi),\ k \in \mathbb{Z}$ ($z = re^{i\theta}$)

Properties:

$$e^{\log(z)} = e^{\ln|r| + i(\theta + 2\pi k)} = e^{\ln|r|}e^{i\theta}e^{i2\pi k} = re^{i\theta} = z$$

$\log(z)$ is not a function but a multi-valued function!

$$\log(e^z) = \log(e^x e^{iy}) = \ln|e^x| + i(y + 2\pi k)$$

$$= x + i(y + 2\pi k)$$

So $\log(z)$ and e^z are not inverse!

Panel 13

Def: The Principle Value of the logarithm is

$$\mathrm{Log}(z) = \ln(r) + i\theta \quad \text{if } z = re^{i\theta}$$

Properties

Log is a function

$e^{\mathrm{Log}(z)} = z = \mathrm{Log}(e^z)$ ⇒ inverse of e^z

$\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z}$ HW use polar form of CR

Recall: $\arg(z)$ and $\mathrm{Arg}(z)$

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Panel 14

Ex: $\ln(x)$ is defined for all $x > 0$

Find $\log(-1)$ and $\mathrm{Log}(-1)$ $\quad \mathrm{Log}(1) = 0$

$\log(-1) = \ln(r) + i(\theta + 2\pi k)$, $\quad -1 = re^{i\theta} = 1e^{i\pi}$

$= \ln(1) + i(\pi + 2\pi k)$

$\Rightarrow \log(-1) = i(\pi + 2\pi k)$

$\mathrm{Log}(-1) = i\pi \qquad (k=0)$

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Panel 15

Derivative of $\log(z)$: HW

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Panel 16

Trig Functions:

Recall: $e^{it} = \cos(t) + i\sin(t)$

$e^{-it} = \cos(-t) + i\sin(-t)$

$= \cos(t) - i\sin(t)$

$\cos(t) = \frac{1}{2}(e^{it} + e^{-it})$

$\sin(t) = \frac{1}{2i}(e^{it} - e^{-it})$

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Panel 17

Complex Trig Functions

Def:

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$\frac{1}{i} = -i$

Ex:

$$\frac{d}{dz}\cos(z) = \frac{1}{2}\left(i e^{iz} - i e^{-iz}\right) = \frac{i}{2}\left(e^{iz} - e^{-iz}\right) = -\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$= -\sin(z)$$

$$\frac{d}{dz}\sin(z) = \cos(z)$$

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Panel 18

Something to Ponder over the Break

$\cos(i)$ easy $\quad \log(i)$ easy

$\sin(i)$ easy

Have a nice, fun-filled break

$i^2 = -1$

$i^{1/2} = 2$ roots

$2^i = ?$

i^i $\Leftarrow$ hint: $a^x = e^{\ln(a^x)} = e^{x \ln(a)}$

i^{i^i}

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