Panel 1

Last Time

$\mathbb{C}$ - diffble: $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$

$\mathbb{C}$ - diffble $\Rightarrow$ continuity

not converse, i.e. $f(z) = \bar{z}$ or $|z|$

Functions in z are usually $\mathbb{C}$-diffble and usual rules (product, quotient, chain rule) apply.

if f is $\mathbb{C}$-diffble, then

CR Equations $u_x = v_y$ $u_y = -v_x$ converse is true if all partials are cont.

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Panel 2

Complex Homework

(1) Prove that $\lim_{z \to i} 3z + 1 = 3i + 1$

(*) (2) Show that $\lim_{z \to 0} \left(z / \bar{z} \right)^2$ does not exist.

Hint: Try $(x,0) \to (0,0)$, $(0,y) \to (0,0)$ and $(x,x) \to (0,0)$

(3) Prove that for $f(z) = z^3$ we have: $f'(z) = 3z^2$

Hint: Factor $z^3 - z_0^3$

(*) (4) Prove that if $f(z_0) = g(z_0) = 0$ and $f'(z_0)$, $g'(z_0)$ exist with $g'(z_0) \neq 0$ then $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z)}{g'(z)}$

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Panel 3

2. Show that $\lim_{z \to 0} \left(\frac{z}{\bar{z}}\right)^2$ DNE $\quad z = x + iy$, $\bar{z} = x - iy$

$\lim_{(x,y)\to(0,0)} \left(\frac{x+iy}{x-iy}\right)^2 \quad \lim_{(x,y)\to(x_0,y_0)} \frac{x^2 + 2xiy - y^2}{x^2 - 2xiy + y^2}$

1\. Let $x = 0$, $y \to 0$: $\lim_{y\to 0} \left(\frac{0+iy}{0-iy}\right)^2 = 1$

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Panel 4

2\. Let $y = y_0$, $x \to x_0$: $\lim_{x\to x_0} \left(\frac{(x_0)^2 + 2x_0 i y_0 - (y_0)^2}{(x_0)^2 - 2x_0 i y_0 + (y_0)^2}\right)$

$= \frac{(x_0)^2 + 2x_0 i y_0 - (y_0)^2}{(x_0)^2 - 2x_0 i y_0 + (y_0)^2})$

3\. Let $y = x$, $x \to x_0$: $\lim_{x\to x_0} \left(\frac{x+iy}{x-iy}\right)^2 \quad \left(\frac{y+iy}{y-iy}\right)^2$;

$\left(\frac{1+i}{1-i}\right)^2 \neq 1$

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Panel 5

$$\underbrace{\frac{f'(z_0)}{g'(z_0)}} \left(= \lim_{z\to z_0} \frac{f'(z)}{g'(z)} \right) = \frac{\displaystyle\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}}{\displaystyle\lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}} =$$

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{g(z)-g(z_0)} = \lim_{z\to z_0} \underline{\frac{f(z)}{g(z)}}$$

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Panel 6

Complex HW

① Show that the following functions are not $\mathbb{C}$-diff'ble:

a) $f(z) = 2x + ixy^2$ b) $f(z) = z - \bar{z}$

c) $f(z) = e^x e^{-iy}$ Hint: check CR equations

② Use the CR equations to show that $f'(z)$ exists if $f(z) = z^3$ and verify that $f'(z) = 3z^2$

(*) ③ Let $f(z) = x^3 + i(1-y)^3$. Show that f is $\mathbb{C}$-diff'ble only for $z = i$ and find $f'(z)$

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Panel 7

$$f(z) = \underbrace{x^3}_{u} + i\,\underbrace{(1-y)^3}_{v} \quad \text{is } \mathbb{C}\text{-diff'ble.}$$

$\Rightarrow$ CR must be true: $u_x = v_y$

$u_y = -v_x$

positive — negative

$$u_x = 3x^2 = v_y = -3(1-y)^2 \qquad \boxed{x=0},\ \boxed{y=1} \Rightarrow \underline{z = i}$$

$x+iy$

$$u_y = 0 \Rightarrow -v_x = 0 \ \checkmark$$

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Panel 8

CR equations in Polar Coordinates

Recall chain rule in $\mathbb{R}$: $\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$

$$\frac{d}{dt} f(x(t)) = \frac{df}{dx} \cdot \frac{dx}{dt}$$

Chain Rule in $\mathbb{R}^2$: $x = x(s,t)$, $y = y(s,t)$, $f(x,y)$

$$\Rightarrow \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

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Panel 9

Ex: $f(x,y) = x y^2$, $x = r\cos(t)$, $y = r\sin(t)$

① $f(r,t) = (r\cos t)(r\sin t)^2 = r^3\cos(t)\sin^2(t)$

$\frac{\partial f}{\partial r} = 3r^2\cos(t)\cdot\sin^2(t)$

$\frac{\partial f}{\partial t} = -r^3\sin^3(t) + 2r^3\cos^2(t)\sin(t)$

② $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = y^2\cos(t) + 2xy\sin(t) =$

$= r^2\sin^2(t)\cos(t) + 2r^2\cos(t)\sin^2(t) =$

$= 3r^2\sin^2(t)\cos(t)$

$\frac{\partial f}{\partial t}$ at HW

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Panel 10

Suppose $f(z) = u(x,y) + iv(x,y)$ and

CR: $u_x = v_y$, $u_y = -v_x$ are true

If $z = re^{it} \Leftrightarrow x = r\cos(t)$, $y = r\sin(t)$

$r\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = r\overset{v_y}{u_x}\cos(t) \mp r\overset{v_x}{u_y}\sin(t)$ $\Rightarrow$ $\boxed{r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial t} \quad \frac{\partial u}{\partial t} = -r\frac{\partial v}{\partial r}}$

$\frac{\partial u}{\partial t} = u_x\frac{\partial x}{\partial t} + u_y\frac{\partial y}{\partial t} = -u_x r\sin(t) + u_y r\cos(t)$

$\frac{\partial v}{\partial r} = v_x\cos(t) + v_y\sin(t)$

$\frac{\partial v}{\partial t} = -v_x r\sin(t) + v_y r\cos(t)$

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Panel 11

Thm (CR in Polar Coordinates)

If f is $\mathbb{C}$-diff'ble and $z = re^{it}$ then

$$r u_r = v_t \quad \text{and} \quad u_t = -r v_r$$

Conversely, if the partial derivatives are continuous and CR is true then f is diff'ble and

$$f'(z) = \ldots \quad \text{HW}$$

$u_r + i v_r$

$\boxed{\text{not used a lot}}$

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Panel 12

Analytic Functions

We want to expand on the idea of $\mathbb{C}$-diff'ble.

Def. A function $f: D \to \mathbb{C}$ is analytic at a point $z_0 \in D$, D a domain, if f is $\mathbb{C}$-diff'ble at all points in an open nbhd. of z_0

Ex: $f(z) = z^2$ is analytic

$f(z) = \bar{z}$ is not analytic

$f(z) = |z|$ is $\mathbb{C}$-diff'ble if $z=0$ but not analytic

$f(z) = 1/z$ is analytic $\forall z \neq 0$

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Panel 13

Def. If a function $f: \mathbb{C} \to \mathbb{C}$ is analytic in $\mathbb{C}$ then f is called <u>entire</u>.

<u>Ex:</u> $f(z) =$ any polynomial is entire

$$f(z) = \underbrace{\cosh(x)\cos(y)}_{u} + i\,\underbrace{\sinh(x)\sin(y)}_{v} \quad \text{is entire?}$$

$u_x = \sinh(x)\cos(y) = v_y = \sinh(x)\cos(y)$ ✓

$u_y = -\cosh(x)\sin(y) = -v_x = \cosh(x)\sin(y)$ ✓

$\forall (x,y) \in \mathbb{C} \Rightarrow f$ is <u>entire</u>.

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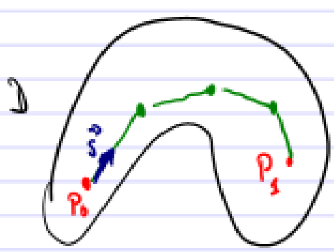
Panel 14

<u>Thm:</u> If f is analytic in a domain D (open, connected) and $f'(z) = 0$ for all $z \in D$, then f is const.

<u>Proof:</u> $f'(z) = u_x + i v_x = 0 \Rightarrow$ $u_x = 0 = v_y = 0$

$u_y = 0 = -v_x = 0$

$\Rightarrow u_x = u_y = v_x = v_y = 0 \ \forall (x,y) \in D$. Take any $P_0, P_1 \in D$. Connect P_0 to P_1 by finitely many line segments. First line segment in direction $\vec{s}$



$$\frac{\partial u}{\partial s} = \operatorname{grad}(u) \cdot \frac{\vec{s}}{\|s\|} = \langle u_x, u_y \rangle \cdot \frac{\vec{s}}{\|s\|} = 0$$

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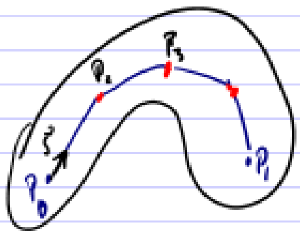
Panel 15



But since $\frac{\partial u}{\partial \xi} = 0$, u must be constant up to point P_2

Repeat: u is constant up to P_3.

Repeat ... u is constant up to P_1

$\Rightarrow u(P_1) = u(P_0)$ for any $P_0, P_1 \in D$

$\Rightarrow u$ is const.

$\Rightarrow f = u + iv$ is constant

Same for v is const.

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Panel 16

Theorem: If $f(z)$ and $\overline{f(z)}$ are both analytic, then f must be constant.

Easy proof: $f = u + iv \Rightarrow$

$$u_x = v_y \quad (1)$$
$$u_y = -v_x \quad (3)$$

$\bar{f} = u + i(-v) \Rightarrow$

$$u_x = -v_y \quad (2)$$
$$u_y = v_x \quad (4)$$

$(1)+(2): \; 2u_x = 0$

$(3)+(4): \; 2u_y = 0$

$\Rightarrow v_x = 0, \; v_y = 0 \Rightarrow f' = 0 \Rightarrow$ f is constant

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Panel 17

Theorem: Suppose $f(z)$ is analytic in a domain D and $|f(z)|$ is constant in D. Then f is constant!

(Compare $f(z) = e^{it} \Rightarrow |f(z)| = |e^{it}| = |\cos(t) + i\sin(t)| = 1$
can't be analytic use CR in polar coordinates)

Proof: $|f| = c \Rightarrow f \cdot \bar{f} = c^2$

If $c = 0 \Rightarrow f = 0 \Rightarrow$ done!

If $c \neq 0$ then $f(z) \neq 0$ and $\overline{f(z)} \neq 0$

But then $\overline{f(z)} = c^2 / f(z)$ is analytic $\Rightarrow f, \bar{f}$ are analytic
$\Rightarrow f$ is const.

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Panel 18

4-5 pm FH #2

Math/CS club meeting

3/14 ~ "π day"

lunch + [illegible]

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