Panel 1

Last time:

$\lim_{z \to z_0} f(z) = L$ given $\varepsilon > 0$ there is $\delta > 0$ s.t. if $|z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$

Problem with limits in $\mathbb{C}$: there are inf. many ways to approach z_0

Thm: If $f(z) = u(x,y) + v(x,y)$ and $L = L_1 + iL_2$ then

$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = L_1$ and $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = L_2$

iff $\lim_{z \to z_0} f(z) = L$

Special limit: $\lim_{z \to z_0} f(z) = f(z_0)$ continuous; $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$ $\mathbb{C}$-diffble

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Panel 2

Results from last time

$\lim_{z \to 1} \frac{iz}{2} = \frac{i}{2}$ proved that ✓ BWO

$\lim_{z \to 0} \frac{\bar{z}}{z}$ = does not exist! $\begin{pmatrix} x = 0 \\ y \to 0 \end{pmatrix}$ or $\begin{pmatrix} x \to 0 \\ y = 0 \end{pmatrix}$ or $\begin{pmatrix} x = y \\ x \to 0 \end{pmatrix}$

$f(z) = z^2 \Rightarrow f'(z) = 2z$

$f(z) = \bar{z} \Rightarrow f'(z)$ = d.n.e. everywhere!!!

Note: Continuity in $\mathbb{C}$ is the same as in $\mathbb{R}^2$ but $\mathbb{C}$-diffble is very different from $\mathbb{R}$ or $\mathbb{R}^2$-diffble

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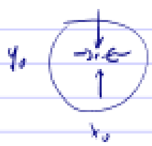
Panel 3

Ex: Let $f(z) = |z|^2$. Is it differentiable? Where?

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0}\frac{z\bar{z} - z_0\bar{z}_0}{z-z_0} = \lim_{(x,y)\to(x_0,y_0)} \frac{x^2+y^2-(x_0^2+y_0^2)}{x+iy-(x_0+iy_0)}$$



$$= \lim_{(x,y)\to(x_0,y_0)} \frac{x^2-x_0^2+(y^2-y_0^2)}{x-x_0+i(y-y_0)}$$

$x = x_0$, $y \to y_0$: $\lim_{y\to y_0} \frac{y^2-y_0^2}{i(y-y_0)} = \frac{2}{i} y_0 = \underline{-i2y_0}$ $\qquad \frac{1}{i} = -i$

must agree only if $\underline{x_0 = y_0 = 0}$

$y = y_0$, $x \to x_0$: $\lim \frac{x^2-x_0^2}{x-x_0} = 2x_0$

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Panel 4

Summary so far:

$f(z) = z^3 \Rightarrow f'(z) = 3z^2$ diffble as you think

$f(z) = \bar{z} \Rightarrow$ diffble nowhere!

- $f(z) = |z|^2 \Rightarrow$ diffble only at $z=0$

Note: If you wrote any of the above functions in terms of $u(x,y)$, $v(x,y)$, then u and v had partials of any order.

- $f(z) = |z|^2 = x^2+y^2 \Rightarrow \frac{\partial}{\partial x} u(x,y) = u_x = 2x$, $u_y = 2y$

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Panel 5

Clearly continuity does not imply $\mathbb{C}$-diffble.

Thm: If f is $\mathbb{C}$-diffble at z_0 then f is continuous there.

Proof $\lim_{z \to z_0} f(z) - f(z_0) \cdot \frac{z-z_0}{z-z_0} = \lim_{z \to z_0} \left(\frac{f(z)-f(z_0)}{z-z_0} \right) (z-z_0) =$

$= f'(z_0) \cdot 0 = 0$ q.e.d.

Geometric Interpretation of $\mathbb{C}$-diffble: ?

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Panel 6

Good news:

$\frac{d}{dz} c = 0$

only works for $z = x+iy$

$\frac{d}{dz} z = 1$

e.g. $f(z) = x^2 y + iy$

$\frac{d}{dz} z^n = n z^{n-1}$

$\frac{d}{dz} (f(z) \pm g(z)) = f'(z) \pm g'(z)$

$\frac{d}{dz} (f(z) \cdot g(z)) = f'(z) \cdot g(z) + f(z) \cdot g'(z)$

$\frac{d}{dz} \frac{f(z)}{g(z)} = \frac{f' g - f g'}{g^2}$

$\frac{d}{dz} f(g(z)) = f'(g(z)) \cdot g'(z)$

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Panel 7

Ex: $f(z) = (2z^2+i)^5$. Find $f'(z)$

$$f'(z) = 5(2z^2+i)^4 \cdot 4z$$

Bad news: function with $\bar{z}$ are generally not differble!

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Panel 8

Cauchy - Riemann Equations:

Suppose $f(z)$ is $\mathbb{C}$-diffble at z_0. Then

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \text{ exists}$$

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} \frac{u(x,y)+iv(x,y)-(u(x_0,y_0)+iv(x_0,y_0))}{x+iy-(x_0+iy_0)} =$$

$$\lim \frac{u(x,y)-u(x_0,y_0)}{x-x_0+i(y-y_0)} + i \lim \frac{v(x,y)-v(x_0,y_0)}{x-x_0+i(y-y_0)} \quad \text{exists}$$

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Panel 9

$$f'(z_0) = \lim_{(x,y)\to(x_0,y_0)} \frac{u(x,y)-u(x_0,y_0)}{x-x_0+i(y-y_0)} + i\,\frac{v(x,y)-v(x_0,y_0)}{x-x_0+i(y-y_0)}$$

let $x \to x_0$, $y = y_0$: $$f'(z) = \lim_{x\to x_0} \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i\lim_{x\to x_0}\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}$$

$$= u_x(x_0,y_0) + i\,v_x(x_0,y_0)$$

Thus: $$f'(z) = \frac{\partial}{\partial x} u + i \frac{\partial}{\partial x} v$$

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Panel 10

$$f'(z_0) = \lim_{(x,y)\to(x_0,y_0)} \frac{u(x,y)-u(x_0,y_0)}{x-x_0+i(y-y_0)} + i\,\frac{v(x,y)-v(x_0,y_0)}{x-x_0+i(y-y_0)}$$

let $x = x_0$, $y \to y_0$: $$\lim_{y\to y_0} \frac{u(x_0,y)-u(x_0,y_0)}{i(y-y_0)} + \cancel{i}\lim_{y\to y_0}\frac{v(x_0,y)-v(x_0,y_0)}{\cancel{i}(y-y_0)}$$

$1/i = -i$

$$= -i\,u_y + v_y$$

Thus: $$f'(z) = -i\frac{\partial}{\partial y} u + \frac{\partial}{\partial y} v$$

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Panel 11

We proved that: if $f(z)$ is $\mathbb{C}$-diffble, then

$$f'(z) = u_x + i v_x$$

$$= -i u_y + v_y$$

<u>Thus:</u>

$$u_x = v_y$$

$$u_y = -v_x$$

CR = <u>Cauchy - Riemann</u> <u>equations</u>

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Panel 12

<u>Thm:</u> If $f(z) = u(x,y) + i v(x,y)$ is $\mathbb{C}$-diffble at a point, then the <u>Cauchy-Riemann</u> equations

$$u_x = v_y \qquad u_y = -v_x \qquad \text{CR}$$

must hold. Moreover:

$$f'(z) = u_x + i v_x$$

$$\left(= -i u_y + v_y \right)$$

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Panel 13

Ex: Suppose $f(z) = z^2$. Show that CR hold.

$u_x =$ $v_y =$

$u_y =$ $-v_x =$

$f(z) = z^2 = \underbrace{x^2 - y^2}_{u(x}+ 2i\underbrace{xy}$

$v(x,y) = 2xy$

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Panel 14

Ex: Let $f(z) =$ random combo of $x^n, y^n, i, \#$

Is it $\mathbb{C}$-diffble?

$\to$ check u, v, then CR

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Panel 15

Thm: If $f(z) = u(x,y) + iv(x,y)$ is defined in a nbhd of $z_0 = x_0 + iy_0$, and

(a) u_x, u_y, v_x, v_y exist and are continuous.

(b) CR hold i.e. $u_x = v_y$

$u_y = -v_x$

Then f is $\mathbb{C}$ diffble and $f'(z_0) = u_x + iv_x$

Proof - Reading

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Panel 16

Ex: Show that the exp. function $f(z) = e^z$ is diff ble and find its derivative.

$e^z = e^{x+iy} = e^x e^{iy} = \underbrace{e^x \cos(y)}_{u} + i\underbrace{e^x \sin(y)}_{v}$

$u_x = e^x \cos(y)$ $\quad v_x = e^x \sin(y)$ $\quad$ CR check ✓

$u_y = -e^x \sin(y)$ $\quad v_y = e^x \cos(y)$

$\Rightarrow f$ is diffble and

$f'(z) = u_x + iv_x = e^x \cos(y) + i e^x \sin(y) =$

$= e^x e^{iy} = e^z = f(z)$

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