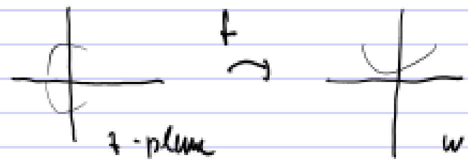
Panel 1

Last time:

Curves: circles $z(t) = z_0 + re^{it}$ and lines $z(t) = at + b$

Complex functions

$\Rightarrow$



Mapping properties of

$f(z) = z^2$

$f(z) = e^z$

$f(z) = 1/z$

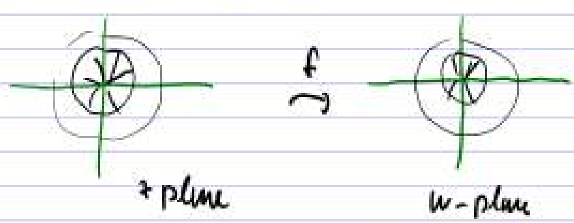
$f(z) = az + b$

Topology: definitions

1

Panel 2

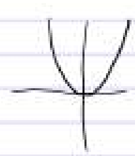
Recall: To visualize the graph of $f: \mathbb{C} \to \mathbb{C}$ we view domain and range next to each other:



e.g. $f(z) = e^z$

How would this approach work for $f: \mathbb{R} \to \mathbb{R}$,

$f(x) = x^2$:



2

Panel 3

Review Questions

① Find $f(1+i)$ for (a) $f(z) = z + z^{-1} + 5$ (b) $f(z) = 4z\bar{z}$

② Let $f(z) = z^{21} - 5z^{7} + 9z^{4}$. Find $f(-1+i)$

③ Express in the form $f(z) = u(x,y) + iv(x,y)$ for

(a) $f(z) = z^3$ (b) $f(z) = \bar{z}^2 + (2-3i)z$

④ Let $f(z) = (3+4i)z - 2+i$. Find the image of

even

(a) the line $x(t) = t,\ y(t) = 1-2t$

(b) the image of the disk $|z-1| < 1$

⑤ Let $f(z) = (2+i)z - 2i$. Find image of the triangle with vertices $z_1 = -2+i,\ z_2 = -2+2i,\ z_3 = 2+i$

3

Panel 4

1. Sketch the following sets and determine which are domains:

(a) $|z-2+i| \le 1$; (b) $|2z+3| > 4$;
(c) $\operatorname{Im} z > 1$; (d) $\operatorname{Im} z = 1$;
(e) $0 \le \arg z \le \pi/4\ (z \ne 0)$; (f) $|z-4| \ge |z|$.

2. Which sets in Exercise 1 are neither open nor closed?

3. Which sets in Exercise 1 are bounded?

7. Determine the accumulation points of each of the following sets:

(a) $z_n = i^n\ (n = 1, 2, \ldots)$; (b) $z_n = i^n/n\ (n = 1, 2, \ldots)$;
(c) $0 \le \arg z < \pi/2\ (z \ne 0)$; (d) $z_n = (-1)^n(1+i)\dfrac{n-1}{n}\ (n = 1, 2, \ldots)$.

8. Prove that if a set contains each of its accumulation points, then it must be a closed set.

even

4

Panel 5

$$|z-4|^2 = |z|^2$$

$$(z-4)(\overline{z-4}) = z\bar{z}$$

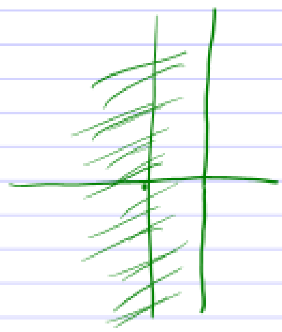
$$(z-4)(\bar{z}-4) = z\bar{z}$$

$$\cancel{z\bar{z}} - 4\bar{z} - 4z + 16 = \cancel{\bar{z}z}$$

$$z + \bar{z} = 4$$

$$2\,\mathrm{Re}(z) = 4$$

$$\mathrm{Re}(z) = 2$$



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Panel 6

A point $z \in D \subset \mathbb{C}$ is called <u>interior point</u> of D if there is an ε-nbhd. of z containing only points of D

A point $z \in \mathbb{C}$ is called <u>exterior point</u> of D if there is an ε-nbhd. of z containing no points from D

A point $z \in \mathbb{C}$ is called a <u>boundary point</u> of D if it is neither interior nor exterior point of D

A set $D \subset \mathbb{C}$ is <u>closed</u> if it contains all of its boundary points.

A point $z_0 \in \mathbb{C}$ is an <u>accumulation point</u> of D if each deleted nbhd. of z_0 contains at least one point of D

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Panel 7

Limits and Continuity:

Recall: $\lim_{x \to x_0} f(x) = L$ means:

for every $\varepsilon > 0$ there is $\delta > 0$ s.t.

if $|x - x_0| < \delta$ $\quad |f(x) - L| < \varepsilon$

In $\mathbb{C}$: $\lim_{z \to z_0} f(z) = L$ means: for every $\varepsilon > 0$ there is a $\delta > 0$ such that

if $|z - z_0| < \delta$ then $|f(z) - L| < \varepsilon$

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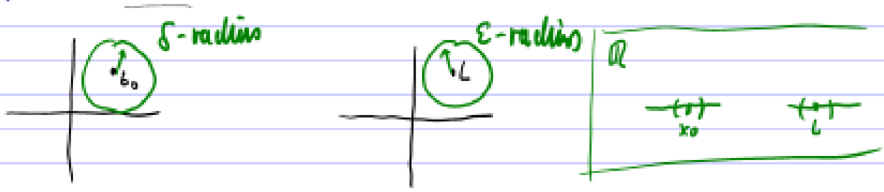
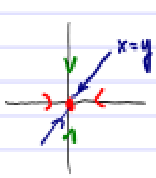
Panel 8

The Trouble with Limits in $\mathbb{C}$: $\lim_{z \to z_0} f(z) = L$ means

$|z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$



$\mathbb{R}^2$: Let $u(x,y) = \frac{xy}{x^2+y^2}$. u has no limit as $(x,y) \to (0,0)$



① Let $y=0$, $x \to 0$: $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x \to 0} \frac{0}{x^2} = \underline{0}$

② Let $x=y \to 0$: $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x \to 0} \frac{x^2}{2x^2} = \underline{\tfrac{1}{2}}$

different

Thus, limit does not exist!

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Panel 9

Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be a complex function defined in a nbhd. of $z_0 = (x_0, y_0)$. Then

$$\lim_{z \to z_0} f(z_0) = w_0 = u_0 + iv_0$$

iff $\lim\limits_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 = u(x_0,y_0)$ and

$$\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0 = v(x_0,y_0)$$

Means: $\mathbb{C}$-limits are exactly the same as $\mathbb{R}^2$-limits, just twice

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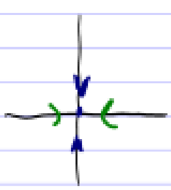
Panel 10

Ex: $f(z) = z / \bar{z}$. Then $\lim\limits_{z \to 0} f(z)$ does not exist:

$$\lim_{(x,y) \to (0,0)} \frac{x+iy}{x-iy}$$



Let $x = 0$, $y \to 0$: $\lim\limits_{y \to 0} \frac{x+iy}{x-iy} = \lim\limits_{y \to 0} \frac{iy}{-iy} = -1$

Let $y = 0$, $x \to 0$: $\lim \frac{x+iy}{x-iy} = \lim\limits_{x \to 0} \frac{x}{x} = 1$

Different, so no limit!

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Panel 11

Ex: $f(z) = i\bar{z}/2$ Then $\lim_{z \to 1} f(z) =$

Proof: if $|z-1| < \delta$ then $|f(z) - \frac{i}{2}| < \varepsilon$

$|i\frac{\bar{z}}{2} - \frac{i}{2}| < \varepsilon$

$|\frac{i}{2}||\bar{z}-1| < \varepsilon$

$|\bar{z}-1| < 2\varepsilon$

$|\overline{z-1}| < 2\varepsilon$

$|z-1| < 2\varepsilon$

scratch paper.

take any $\varepsilon > 0$. Pick $\delta = 2\varepsilon$. Then if $|z-1| < \delta \Rightarrow$

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Panel 12

Def: f a complex function defined in a nbhd of z_0. Then f is continuous at z_0 if

(1) $f(z_0)$ is defined

(2) $\lim_{z \to z_0} f(z)$ exists

(3) $\lim_{z \to z_0} f(z) = f(z_0)$

same as in $\mathbb{R}$ but limits are more complicated

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Panel 13

Continuity Theorems

- A polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ is continuous
- A rational function $r(z) = p_n(z) / q_m(z)$ is cont. where defined
- The sum of two continuous functions is continuous
- The product of two continuous functions is continuous
- The quotient of two continuous functions is continuous where defined.

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Panel 14

Ex: Find $\lim\limits_{z \to 1+i} \frac{z^2 - 2i}{z^2 - 2z + 2} = 0/0$ so bad news

$(1+i)^2 - 2i = 1 + 2i - 1 - 2i = 0$

$(1+i)^2 - 2(1+i) + 2 = 2i - 2 - 2i + 2 = 0$

$z^2 - 2i = (z - \sqrt{2i})(z + \sqrt{2i}) = (z - (1+i))(z + (1+i))$

$z^2 - 2z + 2 = 0 \qquad z_{1/2} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = \begin{matrix} 1+i \\ 1-i \end{matrix}$

$\lim\limits_{z \to 1+i} \frac{z^2 - 2i}{z^2 - 2z + 2} = \lim\limits_{z \to 1+i} \frac{\cancel{(z - (1+i))}(z + (1+i))}{\cancel{(z - (1+i))}(z - (1-i))} = \underline{\underline{\frac{2+2i}{2i}}}$

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Panel 15

Derivatives For $f(z)$ a complex function defined in a nbhd of z_0 we define

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

If that limit exists, f is called $\mathbb{C}$-differentiable, or $\mathbb{C}$-diff'ble.

Ex: $f(z) = z^2$: $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} =$

$$= \lim_{z \to z_0} \frac{\cancel{(z - z_0)}(z + z_0)}{\cancel{(z - z_0)}} = 2z_0$$

$\rightarrow f'(z) = 2z$

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Panel 16

Ex: Show that $f(z) = \bar{z}$ is nowhere diff'ble.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} =$$

$\boxed{\text{! interesting !}}$

$$= \lim_{(x,y) \to (x_0,y_0)} \frac{x - iy - (x_0 - iy_0)}{x + iy - (x_0 + iy_0)} =$$

$$= \lim_{(x,y) \to (x_0,y_0)} \frac{(x - x_0) - i(y - y_0)}{(x - x_0) + i(y - y_0)} = \text{d.n.e.}$$

① let $x = x_0$, $y \to y_0$: $\lim \frac{-i(y - y_0)}{+i(y - y_0)} = -1$

② let $y = y_0$, $x \to x_0$: $\lim \frac{x - x_0}{x - x_0} = 1$

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