Panel 1

Last Time: Root Day

$z^n = a = re^{i\theta}$ always has n different solutions

Algebraically: $z_k = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)}$, $k = 0, 1, 2, \ldots n-1$

Geometrically: n n-th roots are symmetric and form a regular polygon with n sides starting at the principal angle θ/n

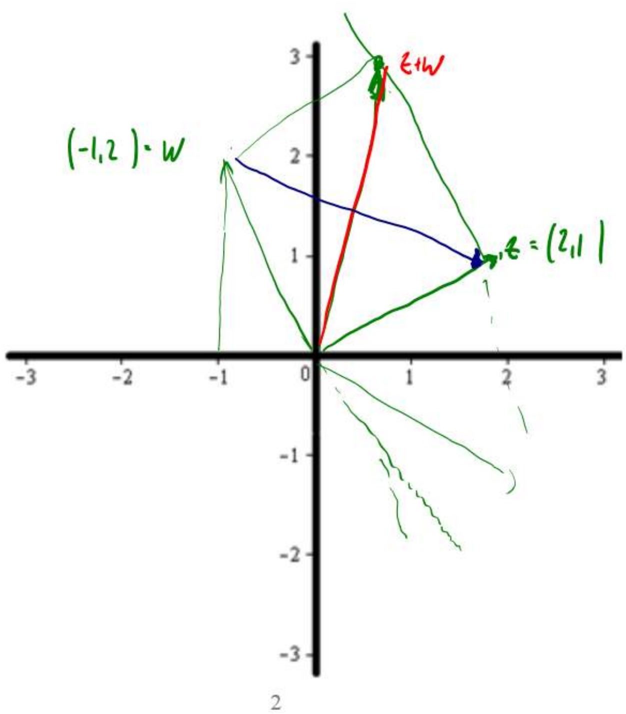
Roots of Unity: $z^n = 1 = 1e^{i0}$, $\omega_k = e^{i\left(\frac{2\pi k}{n}\right)}$, $k = 0, 1, \ldots n-1$

Complex Functions: $f(z) = u(x,y) + i\,v(x,y)$ ✓

1

Panel 2

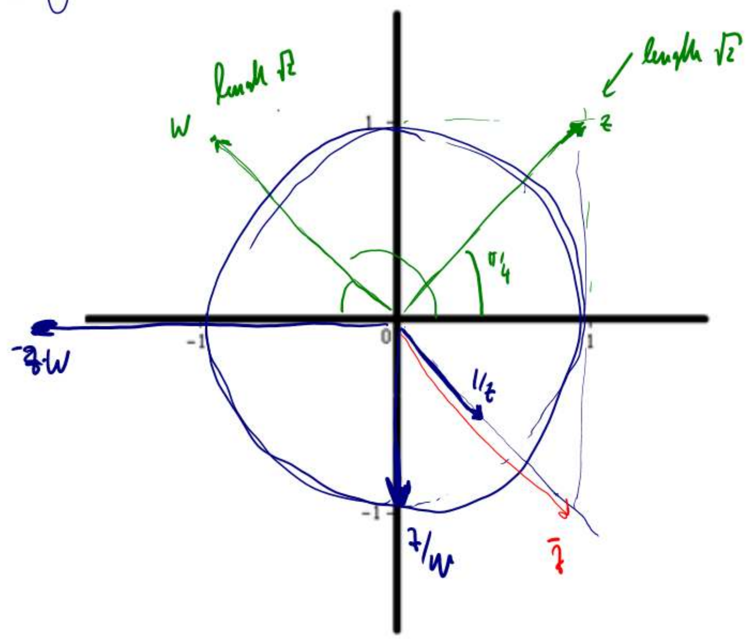
Consider $z = 2 + i$ and $w = -1 + 2i$. Draw the vectors $z, w, z + w$, and $z - w$



2

Panel 3

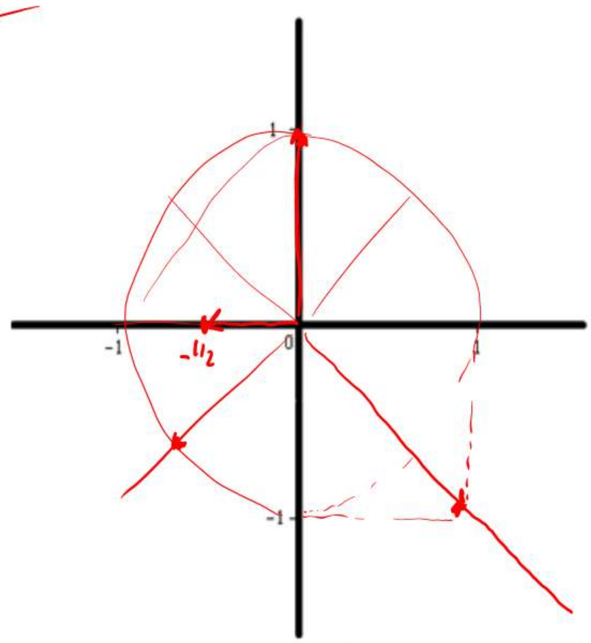
Consider $z = 1 + i$ and $w = -1 + i$. Draw the vectors $z, w, z \cdot w, \frac{z}{w}, \frac{1}{z}$, and $\bar{z}$



3

Panel 4

Draw the following vectors: $z_1 = e^{\frac{i\pi}{2}}, z_2 = 0.5e^{i\pi}, z_3 = \sqrt{2}e^{\frac{-i\pi}{4}}$, and $z_4 = e^{\frac{i5\pi}{4}}$



4

Panel 5

Describe in simple geometric terms what happens to a vector z when:

a. it is multiplied by 2

twice as long, same direction

b. it is multiplied by -1

"opposite"

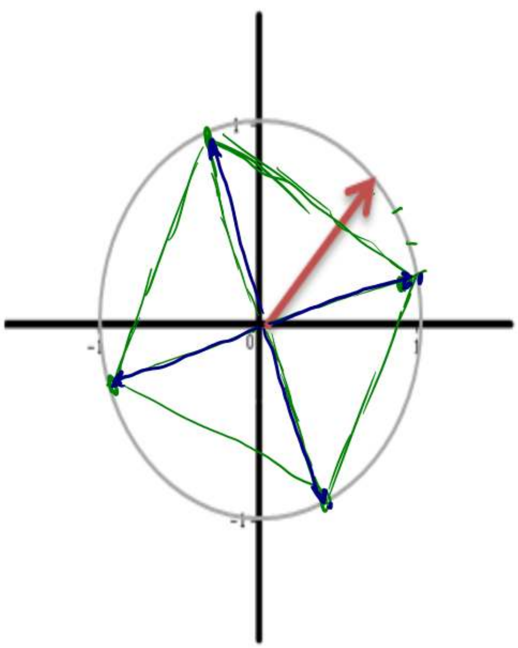
c. it is multiplied by i

rotate by $\pi/2$ $\qquad z \cdot i = (re^{i\theta})(e^{i\pi/2}) = re^{i(\theta+\pi/2)}$

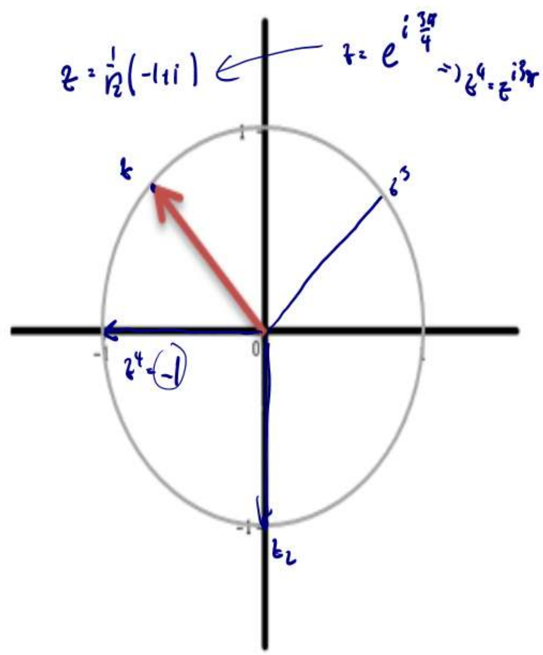
d. it is squared $\qquad (re^{i\theta})^2 = r^2 e^{i2\theta}$

5

Panel 6



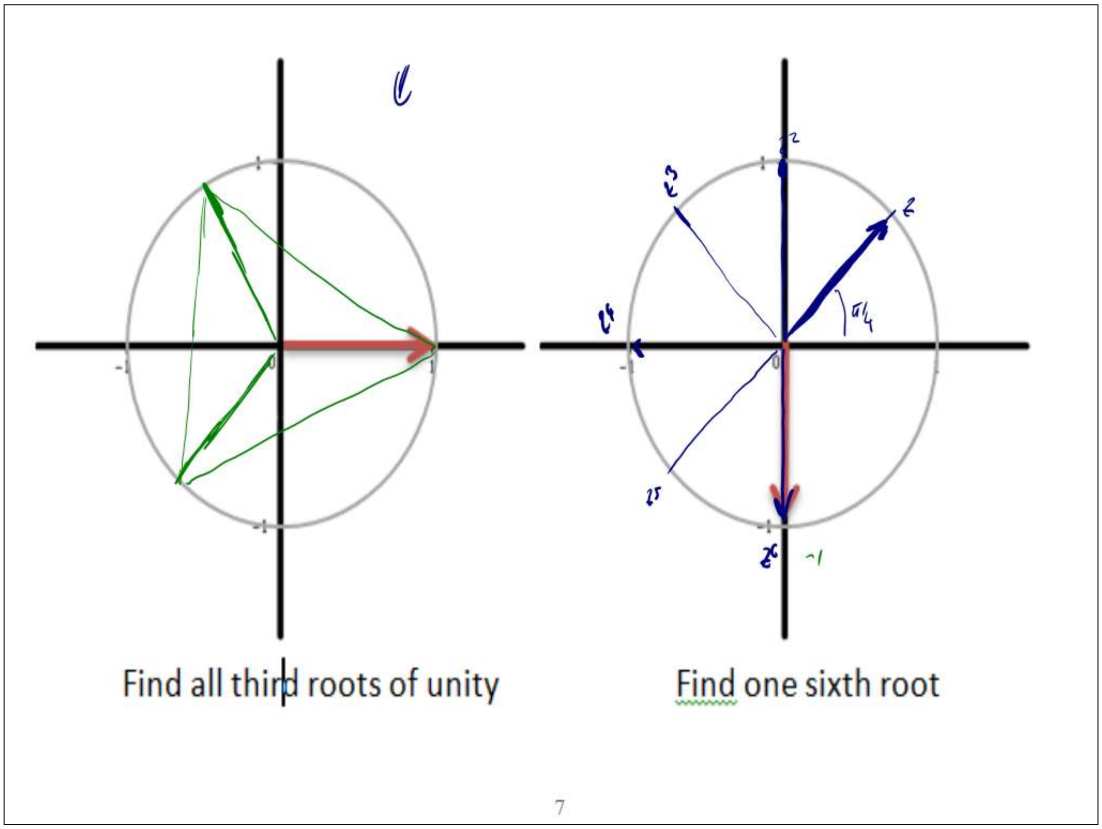
Find all four 4th roots (approx.)



Find z^4

6

Panel 7



Panel 8

Define $\omega_k = e^{i\frac{2\pi k}{n}}$ for $k = 0, 1, 2, \dots, n-1$.

① Show that $(\omega_k)^n = 1$ for $k = 0, 1, 2, \dots, n-1$, i.e. the ω_k are the n-th roots of unity

①

$$(\omega_k)^n = \left(e^{i\frac{2\pi k}{n}}\right)^n = e^{i 2\pi k} = 1 \quad \forall k$$

② Show that $\boxed{(\omega_1)^j = \omega_j}$ for $j = 0, 1, \dots, n-1$

②

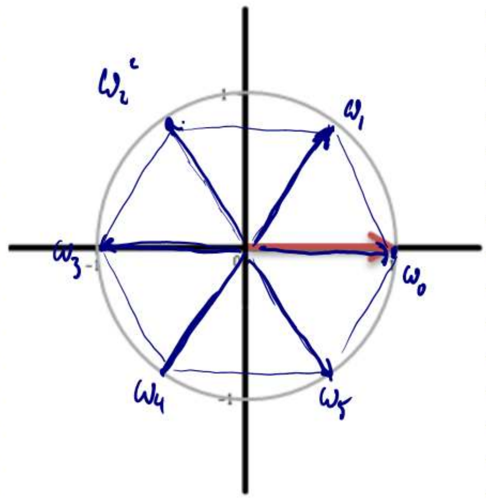
$$(\omega_1)^j = \left(e^{i\frac{2\pi}{n}}\right)^j = \left(e^{i\frac{2\pi j}{n}}\right) = \omega_j$$

8

Panel 9

Define $\omega_k = e^{i\frac{2\pi k}{n}}$ for $k = 0, 1, 2, \ldots, n-1$.

Illustrate $(\omega_k)^n = 1$ and $(\omega_1)^j = \omega_j$ for sixth-roots of 1



$e^{i\frac{2\pi k}{6}}$

9

Panel 10

Theorem: If $\omega_k = e^{i\frac{2k\pi}{n}}$, $k = 0, 1, \ldots, n-1$ are the n-th roots of unity then:

$$1 + \omega_1 + \omega_2 + \omega_3 + \ldots + \omega_{n-1} = 0$$

Lemma: $1 + z + z^2 + z^3 + \ldots + z^{n-1} = \dfrac{1 - z^n}{1 - z}$ $\forall z \neq 1$ (last line)

let $z = \omega_1$:

$$1 + \omega_1 + (\omega_1)^2 + (\omega_1)^3 + \ldots + (\omega_1)^{n-1} = \frac{1 - \overset{1}{\overset{\downarrow}{(\omega_1)^n}}}{1 - z} = 0$$

$$1 + \omega_1 + \omega_2 + \omega_3 + \ldots + \omega_{n-1} = 0$$

10

Panel 11

Thm: Every function $u(x,y)+iv(x,y)$ can be converted to a function $f(z,\bar{z})$

$$x = \frac{1}{2}(z+\bar{z}) \quad \text{and} \quad y = \frac{1}{2i}(z-\bar{z})$$

Ex: $u(x,y)+iv(x,y) = x^2 + iy^2$ ← complex function: $\mathbb{C}\to\mathbb{C}$

$$= \left(\frac{1}{2}(z+\bar{z})\right)^2 + i\left(\frac{1}{2i}(z-\bar{z})\right)^2 =$$

$$= \frac{1}{4}\left(z^2+2z\bar{z}+\bar{z}^2\right) - \frac{i}{4}\left(z^2-2z\bar{z}+\bar{z}^2\right)$$

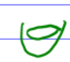
can't get rid of $\bar{z}$

11

Panel 12

Graphs of Complex Functions

$f: \mathbb{R}\to\mathbb{R}$, e.g. $f(x) = x^2$ parabola 2D

$f: \mathbb{R}^2\to\mathbb{R}$ e.g. $f(x,y) = x^2+y^2 = z$  3D

$f: \mathbb{R}^1\to\mathbb{R}^2$ e.g. $f(t) = \langle\cos(t), \sin(t)\rangle$ circle 2D + speed 3D

$\langle\cos(2t), \sin(2t)\rangle$

$f: \mathbb{R}^2\to\mathbb{R}^2$ e.g. $f(x,y) = (x^2-y^2, 2xy)$

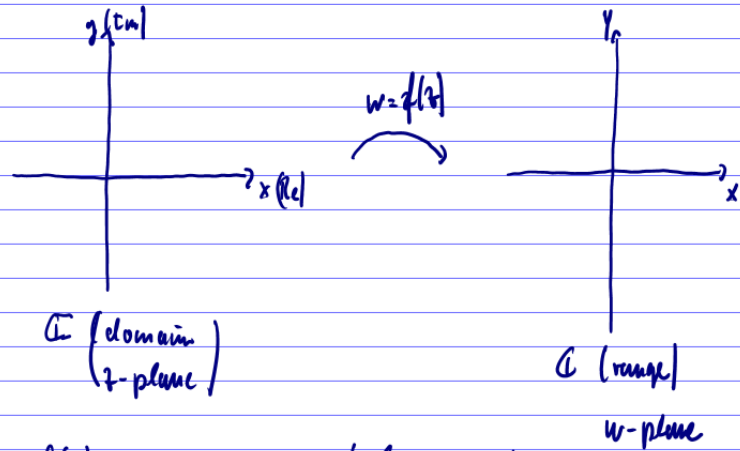
$f: \mathbb{C}\to\mathbb{C}$ $f(z) = z^2$

graphs are 4D objects → no clue!

12

Panel 13

Graphing Complex Functions



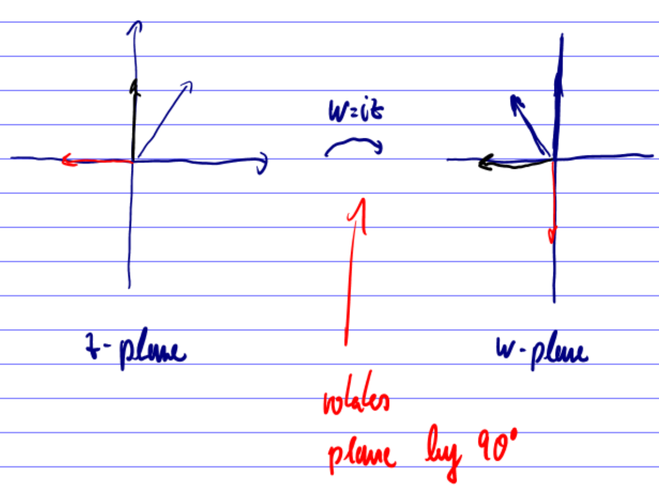
$f(z) = w$ maps $\mathbb{C}$ into $\mathbb{C}$, i.e. it twists and distorts the plane to get another subset of the plane again.

13

Panel 14

Ex: Describe the mapping properties of mult. by i, i.e.

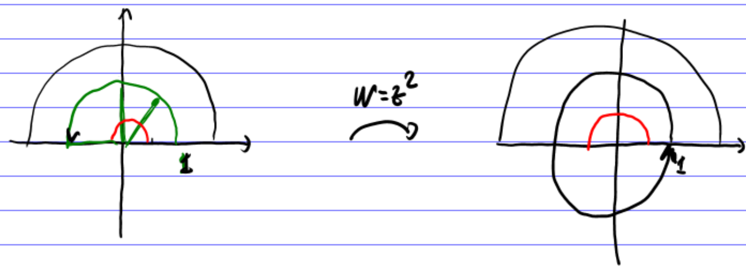
$$f(z) = iz$$



14

Panel 15

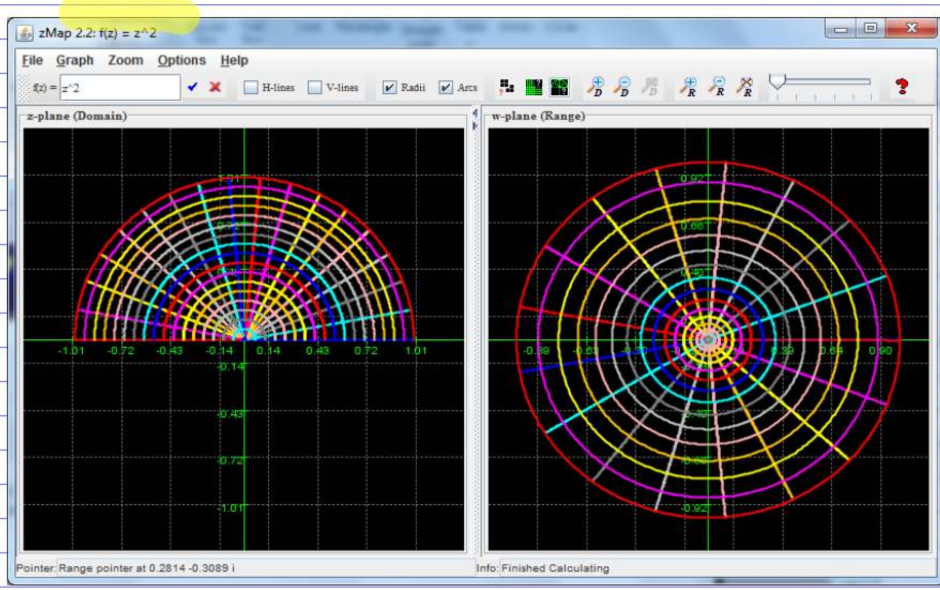
Ex: What does $f(z) = z^2$ do to circles and radii?



15

Panel 16

zMap: Program to visualize complex functions



16

Panel 17

http://www.mathcs.org/java/programs/ZMap/index.html

17