Panel 1

Last time:

$(z^n) = (re^{i\theta})^n = r^n e^{in\theta}$ $\forall n$ integer

Thm. $(e^{i\theta})^n = e^{in\theta}$ $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

de Moivre Formula

Visualized: $z+w,\ z-w,\ z\cdot w,\ z/w,\ z^n,\ 1/z,\ \bar{z}$ visual quiz on Wed

Roots: Solved $\sqrt{i}$

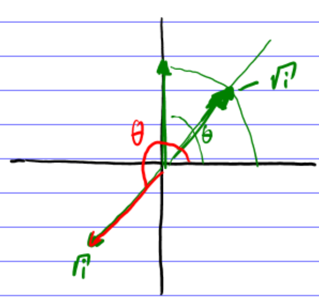
$\sqrt[4]{1}$

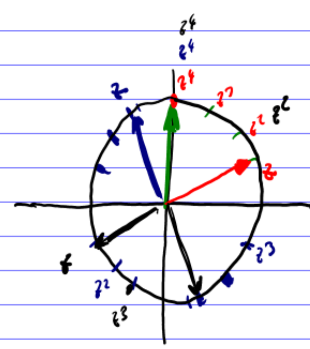
1

Panel 2

Find the given roots graphically:

$\sqrt{i} \Leftrightarrow z^2 = i$ $\sqrt[4]{i} \Leftrightarrow z^4 = i$





2

Panel 3

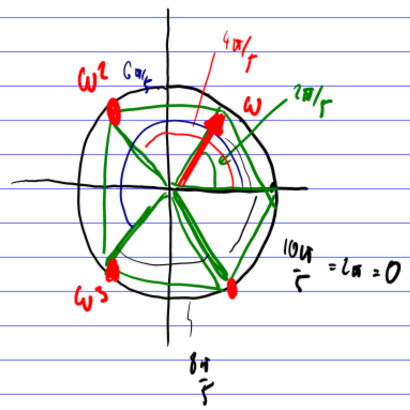
$\omega = e^{i\underbrace{\frac{2\pi}{5}}_{\text{angle}}}$, $|\omega| = |e^{i\frac{2\pi}{5}}| = |\cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5})| =$

$= \sqrt{\cos^2(\frac{2\pi}{5}) + \sin^2(\frac{2\pi}{5})} = 1$

so e^{it} is a unit circle, parametrized

$\omega^0 = 1$ $\qquad$ $\omega^5 = 1$

$\omega = e^{i2\pi/5}$ $\qquad$ $\omega = \sqrt[5]{1}$



$\omega^2 = (e^{i\frac{2\pi}{5}})^2 = e^{i\frac{4\pi}{5}}$

$\omega^3 = (e^{i\frac{2\pi}{5}})^3 = e^{i\frac{6\pi}{5}}$

$\omega^4 = e^{i\frac{8\pi}{5}}$

$\omega^5 = 1$

3

Panel 4

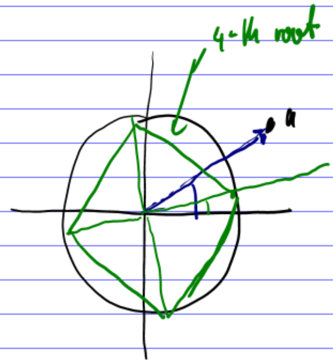
http://math.mit.edu/daimp/ComplexRoots.html

4

Panel 5

Finding roots graphically: $\sqrt[n]{a} = z$ or $z^n = a$

(1) Draw unit circle with a as a vector in polar coordinates



(2) Divide angle by n ⇐ principle angle

(3) Draw a regular polygon with n vertices, starting at the first angle

(4) Adjust the lengths by $\sqrt[n]{r} \in \mathbb{R}$

5

Panel 6

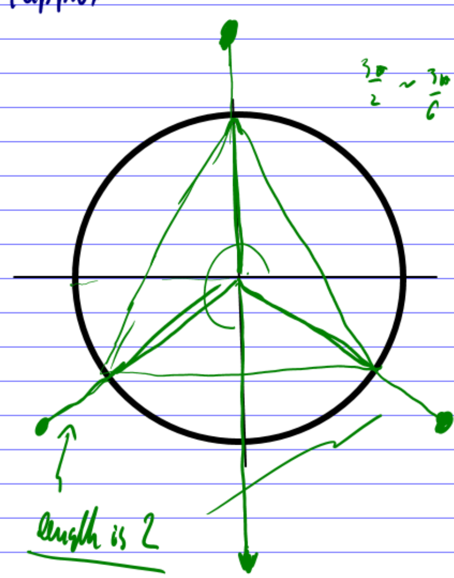
Find $\sqrt[3]{-8i}$ graphically

(1) Draw unit circle and $-8i$, approx

(2) Divide angle by 3

(3) Draw regular polygon with 3 corners

(4) Adjust the radius to be 2



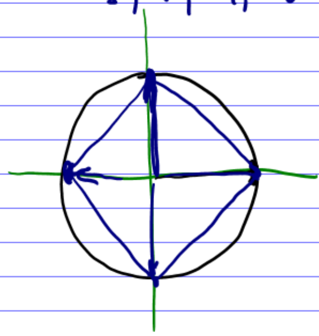
$\frac{3\pi}{2} \sim \frac{3\pi}{6}$

$(-8i)^{1/3} = 2i$ ✓

6

Panel 7

$\sqrt[4]{1} = 1, i, -1, -i$



7

Panel 8

To find n-th root of a algebraically: $z^n = a = re^{i\theta}$

$z^n = re^{i\theta}$	$z^n = re^{i(\theta+2\pi)}$	$z^n = re^{i(\theta+4\pi)}$	
$z = r^{1/n} e^{i\frac{\theta}{n}}$	$z = r^{1/n} e^{i\left(\frac{\theta}{n}+\frac{2\pi}{n}\right)}$	$z = r^{1/n} e^{i\left(\frac{\theta}{n}+\frac{4\pi}{n}\right)}$	...
$\downarrow$	$\downarrow$		
$(z)^n = \left(r^{1/n} e^{i\theta/n}\right)^n$	$(z)^n = \left(r^{1/n} e^{i\left(\frac{\theta}{n}+\frac{2\pi}{n}\right)}\right)^n$		

Theorem: The n-th roots of $a = re^{i\theta}$ are

$$r^{1/n} e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)}, \quad k = 0, 1, 2, \ldots$$

8

Panel 9

Ex: Find all 4th roots of i, $z^4 = i = e^{i\pi/2}$

① $i = e^{i\pi/2} \Rightarrow \sqrt[4]{i} = e^{i\frac{\pi/2}{4}} = e^{i\pi/8}$ check: $\left(e^{i\pi/8}\right)^4 = e^{i\pi/2} = i$ ✓

② $i = e^{i(\pi/2 + 2\pi)} \Rightarrow \sqrt[4]{i} = e^{i\left(\frac{\pi/2}{4} + \frac{2\pi}{4}\right)} = e^{i\frac{5\pi}{8}}$ check $\left(e^{i\frac{5\pi}{8}}\right)^4 = e^{i\frac{10\pi}{2}} = e^{i\pi/2 + 2\pi}$

③ $i = e^{i(\pi/2 + 4\pi)} \Rightarrow \sqrt[4]{i} = e^{i\left(\frac{\pi/2}{4} + \frac{4\pi}{4}\right)} = e^{i\frac{9\pi}{8}}$ ✓

④ $i = e^{i(\pi/2 + 6\pi)} \Rightarrow \sqrt[4]{i} = e^{i\left(\frac{\pi/2}{4} + \frac{6\pi}{4}\right)} = e^{i\frac{13\pi}{8}}$

⑤ $i = e^{i(\pi/2 + 8\pi)} \Rightarrow \sqrt[4]{i} = e^{i\left(\frac{\pi/2}{4} + \frac{8\pi}{4}\right)} = e^{i(\pi/8 + 2\pi)} = e^{i\pi/8}$ already there!

9

Panel 10

Theorem: $z^n = a = re^{i\theta}$ has n different solutions

$$z_i = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, 2, \ldots, n-1$$

because of $k = n$ it repeats

Ex: Find $\sqrt[2]{1}$, $\sqrt[4]{1}$, $\sqrt[3]{-1}$

10

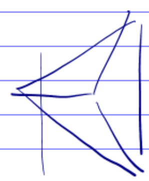
Panel 11

Theorem: $z^n = a = re^{i\theta}$ has n different solutions

$$z_i = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)} \quad , k = 0, 1, 2, \dots n-1$$

$\sqrt[3]{-1}$: $-1 = e^{i\pi}$, 1st root: $e^{i\pi/3}$ $k=0$

2nd root: $e^{i\left(\pi/3 + \frac{2\pi}{3}\right)} = e^{i\pi} = -1$



3rd root: $e^{i\left(\pi/3 + \frac{4\pi}{3}\right)} = e^{i 5\pi/3}$

4th root: $e^{i\left(\pi/3 + \frac{6\pi}{3}\right)}$

11

Panel 12

Theorem: $z^n = a = re^{i\theta}$ has n different solutions

$$z_i = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)} \quad , k = 0, 1, 2, \dots n-1$$

done

12

Panel 13

Find the n-th roots of unity, i.e. $\sqrt[n]{1}$; $1 = e^{i0}$

1st root: $\omega_0 = e^{i0/n} = e^{i0} = 1$

2nd root: $\omega_1 = e^{i\left(0/n + \frac{2\pi}{n}\right)} = e^{i\frac{2\pi}{n}}$

$\omega_2 = e^{i\frac{4\pi}{n}}$

$\vdots$

$\omega_k = e^{i\frac{2\pi k}{n}}$

The n n-th root of unity (1) are $\boxed{\omega_k = e^{i\frac{2\pi k}{n}}, k = 0, 1, \dots n-1}$

http://math.mit.edu/daimp/ComplexRoots.html

13

Panel 14

Theorem: If $\omega_k = e^{i\frac{2k\pi}{n}}$, $k = 0, 1, \dots, n-1$ are the n-th roots of unity then:

$$1 + \omega_1 + \omega_2 + \omega_3 + \dots + \omega_{n-1} = 0$$

Proof: HW

Lemma

$$1 + \cancel{z} + \cancel{z^2} + \cancel{\dots} + \cancel{z^{n-1}} = S_n$$

$$-\left(\cancel{z} + \cancel{z^2} + \cancel{z^3} + \cancel{\dots} + z^n = zS_n\right)$$

$$1 - z^n = S_n - zS_n = S_n(1-z)$$

$$S_n = \frac{1-z^n}{1-z}$$

$$1 + z + z^2 + z^3 + \dots + z^{n-1} = \frac{1-z^n}{1-z}$$

use lemma and $z = \omega_1$ see what you get

14

Panel 15

Complex Functions

A complex function is a rule that assigns to every z in a domain $D \subset \mathbb{C}$ a complex number w.

Ex: $f(z) = z^2$ domain $\mathbb{C}$ $\qquad g(z) = x^2 + y^2$

$f(z) = \frac{5+z}{1+z^2}$ domain $\mathbb{C} - \{\pm i\}$ $\qquad h(z) = z - \bar{z} = 2i\,\mathrm{Im}(z)$

Note: If no domain is specified explicitly, we assume the largest possible subset of $\mathbb{C}$ as domain.

15

Panel 16

Thm: Every complex function $f(z) = w$ can be written as $f(z) = u(x,y) + i v(x,y)$

Ex: $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$

$g(z) = z\bar{z} = |z|^2 = x^2 + y^2$

$u(x,y) = x^2 + y^2, \quad v(x,y) = 0$

$h(z) = ix^2$

$u(x,y) = 0 \qquad v(x,y) = x^2$

16

Panel 17

Thm: Every function $u(x,y)+iv(x,y)$ can be converted to a function $f(z,\bar{z})$

$x = \frac{z+\bar{z}}{2}$

$y = \frac{z-\bar{z}}{2i}$

Ex: $u(x,y)+iv(x,y) = x^2-y^2+2ixy$

$= \frac{1}{4}(z+\bar{z})^2 + \frac{1}{4}(z-\bar{z})^2 + 2i\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right) =$

$= \frac{1}{4}(z^2+\cancel{2z\bar{z}}+\bar{z}^2) + \frac{1}{4}(z^2-\cancel{2z\bar{z}}+\bar{z}^2) + \frac{1}{2}(z^2-\bar{z}^2) =$

$= \underline{\underline{z^2}}$

17

Panel 18

Graphs of Complex Functions

next time

picture quiz next time

18