Panel 1

Last Time:

$z \in \mathbb{C}$ means

$z = (x,y)$ or $x+iy$, $i = (0,1)$

$z_1 + z_2 = (x_1+x_2, y_1+y_2)$

$z_1 \cdot z_2 =$ funky $(x_1+iy_1)\cdot(x_2+iy_2)$, $i^2 = -1$

$\mathbb{R}^2$: $\langle 1,2\rangle + \langle 3,4\rangle = \langle 4,6\rangle$

$\langle 1,2\rangle \cdot \langle 3,4\rangle = 1\cdot 3 + 2\cdot 4 \in \mathbb{R}$

$Re(z) = x$

$Im(z) = y$

z/w

equations

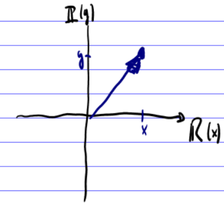
1

Panel 2

Complex Numbers Graphically

$z = (x,y) = x+iy$:

every $z \in \mathbb{C}$ has a vector representation in "$\mathbb{R}^2$"



Def: The absolute value, or norm, or modulus of $z \in \mathbb{C}$ is

$$|z| = \|z\| = \|x+iy\| = \sqrt{x^2+y^2}$$

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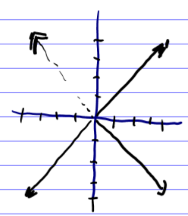
Panel 3

Find $|1+i| = \sqrt{2}$

$|4+4i| = |4(1+i)| = 4\,|1+i| = 4\sqrt{2}$

$|-4-4i| = |-(4+4i)| = 4\sqrt{2}$

$|4-4i| = 4\,|1-i| = 4\sqrt{1+(-1)^2} = 4\sqrt{2}$



Which one is smaller: $z = 1+3i$ or $w = 2+2i$?

$\boxed{\text{Impossible!}}$ $z_1 < z_2$ nonsense but $|z| > |w|$

3

Private Freehand 3

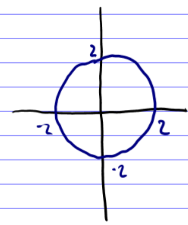
3

Panel 4

Describe the set of all z s.t. $|z| = 2$

$|z| = |x+iy| = \sqrt{x^2+y^2} = 2$

$\Leftrightarrow x^2+y^2 = 4$



circle radius 3 : $|z| = 3$

circle, center $(1,2)$ and radius 4: $|z - (1+2i)| = 4$

$1+2i$

$|x+iy - (1+2i)| = |x-1+i(y-2)|$

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Panel 5

Quick Properties:

If $c \in \mathbb{R}$ and $z \in \mathbb{C}$ then $|c \cdot z| = |c||z|$

If $z, w \in \mathbb{C}$ then $|z \cdot w| = |z| \cdot |w|$

$Re(z) \leq |Re(z)| \leq |z|$

$Im(z) \leq |Im(z)| \leq |z|$

Triangle Inequality:

$|z+w| = |z| + |w|$

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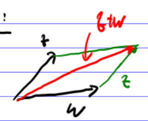
Panel 6

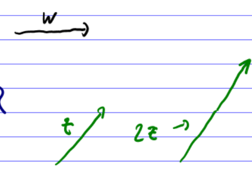
Complex Numbers Graphically:

$|z+w| \leq |z| + |w|$

How to add : $z+w$

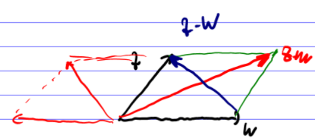


How to find $c \cdot z$, $c \in \mathbb{R}$



How to subtract : $z - w$

$z + (-w)$



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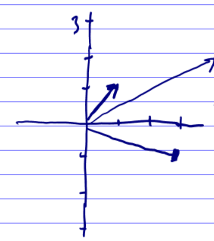
Panel 7

How to multiply: $z \cdot w$

$(1+i)(3-i) = 3+1+i(2) = 4+2i$



?

later

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Panel 8

Complex Conjugate

Def: If $z = x + iy$, then the conjugate of z is

$\bar{z} = \operatorname{conj}(z) = x - iy$

Ex: $\overline{3+4i} = 3-4i$

$\overline{-1-i} = -1+i$

$\overline{(1+i)(1-i)} = (1-i)(1+i)$ ← check! HW

$\frac{z+\bar{z}}{2} = \frac{x+iy+x-iy}{2} = x = \operatorname{Re}(z)$

$\frac{z-\bar{z}}{2i} = \frac{x+iy-(x-iy)}{2i} = \frac{2iy}{2i} = y = \operatorname{Im}(z)$

Panel 9

Properties of Conjugates

(a) $\overline{z+w} = \bar{z} + \bar{w}$

(b) $\overline{z\cdot w} = \bar{z}\cdot\bar{w}$

(c) $\overline{\left(\frac{z}{w}\right)} = \bar{z}/\bar{w}$

(d) $\overline{(\bar{z})} = z$

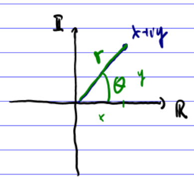
(e) $\operatorname{Re}(z) = \frac{z+\bar{z}}{2}$

(f) $\operatorname{Im}(z) = \frac{z-\bar{z}}{2i}$

(g) $z\cdot\bar{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2$

Panel 10

Still not solved: How to visualize complex multiplication



Instead of $z = x+iy$ we write

$x = r\cos(\theta)$

$y = r\sin(\theta)$

Def: Every complex number $z \neq 0$ has modulus $|z| = r = \sqrt{x^2+y^2}$ and an angle θ called $\arg(z)$ (argument of z) and $z = x+iy = r(\cos(\theta) + i\sin(\theta))$

Def: $\operatorname{Arg}(z)$ is the principle argument of z, i.e. that angle θ with $-\pi < \theta \leq \pi$

Panel 11

Ex: $z = 5 \Rightarrow \arg(z) = 0$ or 2π or 4π

$\operatorname{Arg}(z) = 0$

$z = i \Rightarrow \arg(z) = \pi/2$

$\operatorname{Arg}(z) = \pi/2$

$z = -i \Rightarrow \arg(z) = 3\pi/2$

$\operatorname{Arg}(z) = -\pi/2$

$z = -5 \Rightarrow \arg(z) = \pi$

$\operatorname{Arg}(z) = \pi$

Panel 12

Recall from Calc 2:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Thus: $e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \dots$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \dots$$

$$= \underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots}_{\cos(x)} + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots\right)}_{\sin(x)}$$

$$\cos(x) + i \sin(x)$$

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Panel 13

Euler's Formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

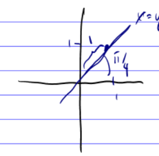
what if $\theta = \pi$: $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$

$\boxed{e^{i\pi} + 1 = 0}$ Euler's Equation

Def: Every $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ $= x + iy$

Ex: $e^{i\pi/2} = i$ $\qquad e^{i\pi/4}$

$re^{i\pi} = -r$ $\qquad \frac{1}{\sqrt{2}}(1+i)$



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Panel 14

Geometrically: $z = Re^{i\theta}$ is the

later

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Panel 15

How does this help visualizing multiplication?

Take $z_1 = R_1 e^{i\theta_1}$ $\quad z_2 = R_2 e^{i\theta_2}$

$z_1 \cdot z_2 = (R_1 e^{i\theta_1}) \cdot (R_2 e^{i\theta_2})$

$$= R_1(\cos(\theta_1) + i\sin(\theta_1)) \cdot R_2(\cos(\theta_2) + i\sin(\theta_2)) =$$

$$= R_1 \cdot R_2 \left(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i\left(\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)\right)\right)$$

$$= R_1 R_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right) =$$

$$= R_1 R_2 e^{i(\theta_1 + \theta_2)}$$

$z_1 \cdot z_2$: mult their radii, and add the angles!

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Panel 16

Ex:

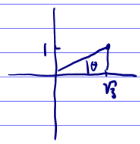
$$i^2 = i \cdot i = -1$$

$$= e^{i\pi/2} \cdot e^{i\pi/2} = e^{i(\pi/2 + \pi/2)} = e^{i\pi} = -1$$

$$(1+i)^4 = (\sqrt{2})^4 e^{i\pi} = -2^2 = \underline{\underline{-4}}$$

$$(\sqrt{3}+i)^7 = [\underbrace{2}\, e^{i\pi/3}]^7 = 2^7 e^{i7\pi/3} = 2^6 e^{i\pi} \underbrace{\cdot 2 e^{i\pi/3}}$$

$$= \underline{\underline{-64 \cdot (\sqrt{3}+i)}}$$



$$\tan(\theta) = 1/\sqrt{3}, \quad \theta = \underline{\underline{\pi/3}}$$

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